Contrôle frontière des équations de Navier-Stokes

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N° d'ordre: 132-2014

THÈSE DE DOCTORAT

Présentée pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ CLAUDE BERNARD LYON 1

et le grade de

DOCTEUR DE L'UNIVERSITÉ GASTON BERGER DE SAINT-LOUIS

Spécialité: Mathématiques appliquées

par

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Soutenue publiquement le 04 Juillet 2014 devant la commission d'examen:

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Résumé

Cette thèse est consacrée à l'étude de problèmes de stabilisation exponentielle par retour d'état ou "feedback" des équations de Navier-Stokes dans un domaine borné $\Omega \subset \mathbb{R}^d$, $d = 2$ ou 3. Le cas d'un contrôle localisé sur la frontière du domaine est considéré. Le contrôle s'exprime en fonction du champ de vitesse à l'aide d'une loi de feedback non-linéaire. Celle-ci est fournie grâce aux techniques d'estimation a priori via la procédure de Faedo-Galerkin laquelle consiste à construire une suite de solutions approchées en utilisant une base de Galerkin adéquate. Cette loi de feedback assure la décroissance exponentielle de l'énergie du problème discret correspondant et grâce au résultat de compacité, nous passons à la limite dans le système satisfait par les solutions approchées. Le chapitre 1 étudie le problème de stabilisation des équations de Navier-Stokes autour d'un état stationnaire donné, tandis que le chapitre 2 examine le problème de stabilisation autour d'un état non-stationnaire prescrit. Le chapitre 3 est consacré à l'étude de la stabilisation du problème de Navier-Stokes avec des conditions aux bords mixtes (Dirichlet-Neumann) autour d'un état d'équilibre donné. Enfin, nous présentons dans le chapitre 4, des résultats numériques dans le cas d'un écoulement autour d'un obstacle circulaire.

Mots-clefs : Système de Navier-Stokes, contrôle feedback, stabilisation frontière, approche de Galerkin.

Abstract

In this thesis we study the exponential stabilization of the two and three-dimensional Navier-Stokes equations in a bounded domain Ω, by means of a boundary control. The Control is expressed in terms of the velocity field by using a non-linear feedback law. In order to determine a feedback law, we consider an extended system coupling the Navier-Stokes equations with an equation satisfied by the control on the domain boundary. While most traditional approaches apply a feedback controller via an algebraic Riccati equation, the Stokes-Oseen operator or extension operators, a Galerkin method is proposed instead in this study. The Galerkin method permits to construct a stabilizing boundary control and by using energy a priori estimation technics, the exponential decay is obtained. A compactness result then allows us to pass to the limit in the nonlinear system satisfied by the approximated solutions. Chapter 1 deals with the stabilization problem of the Navier-Stokes equations around a given steady state, while Chapter 2 examines the stabilization problem around a prescribed non-stationary state. Chapter 3 is devoted to the stabilization of the Navier-Stokes problem with mixed-boundary conditions (Dirichlet-Neumann), around to a given steady-state. Finally, we present in Chapter 4, numerical results in the case of a flow around a circular obstacle.

Keywords : Navier-Stokes system, feedback control, boundary stabilization, Galerkin method.

Dédicaces

A mon cher grand frère feu David Ngor Diouma NGOM.

A mon cher grand frère Paul Gonar NGOM,

Aux membres de ma famille.

A mes collègues de travail.

A tous mes chers amis sans exception.

Remerciements

Je tiens à exprimer ma profonde reconnaissance à mes deux directeurs de thèse, le professeurs Abdou Sène de l'Université Gaston Berger (UGB), Saint-Louis du Sénégal et le professeur Daniel le Roux de l'Université Claude Bernard (UCLB), Lyon 1. Sachez, chers professeurs, que c'est avec beaucoup d'humilité que je vous adresse mes remerciements pour la confiance que vous m'avez accordée, pour votre disponibilité et pour vos précieux conseils.

Je suis très honoré que les professeurs Mehdi Badra (Laboratoire de mathématiques et de leurs applications, université de Pau et des pays de l'Adour) et Viorel Barbu (Institute of mathematics of the Romanian Academy) aient accepté de rapporter cette thèse. Je leur suis très reconnaissant pour l'intérêt qu'ils ont accordé à mon travail.

Je voudrai exprimer toute ma gratitude à Monsieur Jérôme Le Rousseau, professeur à l'université d'Orléans pour l'intérêt qu'il a bien voulu porter à ce travail et avoir accepté de présider mon jury. J'adresse également tous mes remerciements à Monsieur Didier Bresch, professeur à l'Université de Savoie, Monsieur Andro Mikelic, professeur à l'Université Claude Bernard (UCLB), Lyon 1et à Monsieur Mamadou Sy, professeur à l'Université Gaston Berger de Saint-Louis qui ont accepté de faire partie de mon jury.

Je tiens à remercier et à témoigner toute ma reconnaissance aux organisations financières qui ont contribué à la réussite de cette thèse, à savoir, le Service de Coopération et d'Action Culturelle (SCAC) de Ambassade de France à Dakar, aux programmes d'Appuis Intègres pour le Renforcement des Équipes Scientifiques du Sud (AIRES-Sud), au Labex MILYON: Communauté de mathématique et d'informatique fondamentale à Lyon ainsi qu' à l'institut de Recherche pour le Développement (IRD) au Sénégal.

Je remercie l'institut Camille Jordan (IJC) pour l'excellent accueil que son équipe m'a réservé durant mes séjours. Merci aussi d'avoir mis à ma disposition tous les documents scientifiques nécessaires et indispensable à mes travaux de recherches. Je remercie aussi toute l'équipe administrative et technique de L'UFR SAT ainsi que l'équipe du laboratoire d'Analyse Numérique et d'Informatique (LANI).

Je voudrais remercier également tous mes chers amis, sans exception, qui m'ont soutenus durant cette période de recherche. Je remercie aussi tous mes collègues de travail que j'ai eu le plaisir de rencontrer et de côtoyer durant ces années de travail.

Enfin, un grand merci à ma famille, qui m'a soutenue de manière infaillible pendant toutes ces années.

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Introduction Générale

1 Contexte de la thèse

En mécanique des fluides, les équations de Navier-Stokes sont des équations aux dérivées partielles non linéaires qui décrivent le mouvement des fluides « newtoniens » (liquide et gaz visqueux ordinaires) dans l'approximation des milieux continus. Par une résolution approchée, elles permettent de proposer une modélisation des courants océaniques et des mouvements des masses d'air de l'atmosphère pour les océanographes et les météorologistes, la simulation numérique du comportement des gratte-ciel ou des ponts sous l'action du vent pour les architectes et ingénieurs, des avions, trains ou voitures à grande vitesse pour leurs bureaux d'études concepteurs, mais aussi le trivial écoulement de l'eau dans un tuyau et de nombreux autres phénomènes d'écoulement de divers fluides. Le cas particulier de l'écoulement d'un fluide incompressible est traité dans cette thèse. L'écoulement d'un fluide est dit incompressible lorsque l'on peut négliger ses variations de masse volumique au cours du temps. Cette hypothèse est vérifiée pour l'eau liquide et les métaux en fusion.

Plusieurs travaux dédiés à l'étude du système de Navier-Stokes incompressibles ont été effectués dans la littérature (voir par exemple [14, 20, 32]). Ces travaux ont permis d'établir des résultats d'existence, d'unicité et de régularité de la solution dans des domaines bornés ou non bornés, des résultats relatifs au comportement en temps long des solutions, ainsi que des résultats concernant les problèmes fondamentaux de stabilité. Nous mentionnons cependant qu'à l'heure actuelle la question de l'existence globale (c'est-à-dire pour tout temps $t > 0$) de solutions régulières en dimension 3, de même que celle de l'unicité des solutions faibles toujours en dimension 3 sont des questions ouvertes.

Dans cette thèse, nous nous intéressons à l'étude de problèmes de stabilisation par retour d'état ou « feedback » des équations de Navier-Stokes incompressibles, dans un domaine borné, autour d'un état désiré, à l'aide d'un contrôle frontière dynamique. La stabilisation par retour d'état permet de gérer, commander, diriger ou réguler le comportement d'un système physique comme le phénomène d'écoulement autour d'un barrage hydraulique. La construction d'un barrage peut provoquer à la fois des bouleversements humains en forçant des populations entières à se déplacer, et avoir un impact écologique non négligeable en changeant l'écosystème local. Cependant, il permet par exemple la régulation du débit d'une rivière ou d'un fleuve (favorisant ainsi le trafic fluvial), l'irrigation des cultures, une prévention relative des catastrophes naturelles (crues, inondations), par la création de lacs artificiels ou de réservoirs. Un barrage autorise aussi, sous certaines conditions, la production de force motrice (moulin à eau) et d'électricité : on parle alors de barrage hydroélectrique (voir Figure 1). L'énergie électrique est produite par la transformation de l'énergie cinétique de l'eau en énergie électrique par l'intermédiaire d'une turbine hydraulique couplée à un générateur électrique (voir Figure 2 et Figure 3). Pour les barrages au fil de l'eau la quantité d'énergie produite est directement liée au débit (m^3/s, m^3/h, m^3/j, m^3/an). Pour les barrages par accumulation, la quantité d'énergie disponible, sur une période donnée, dépend du volume de son réservoir, des apports et pertes naturels sur la période et de la hauteur de chute. Afin d'augmenter ou de diminuer la quantité d'énergie produite, nous pouvons agir sur les vannes (voir partie E de la Figure 1). Cette action permet de contrôler le débit entrant ou sortant au niveau de la conduite forcée (voir partie F de la Figure 1). On s'intéresse alors au problème de stabilisation par retour d'état des équations de Navier-Stokes incompressibles.

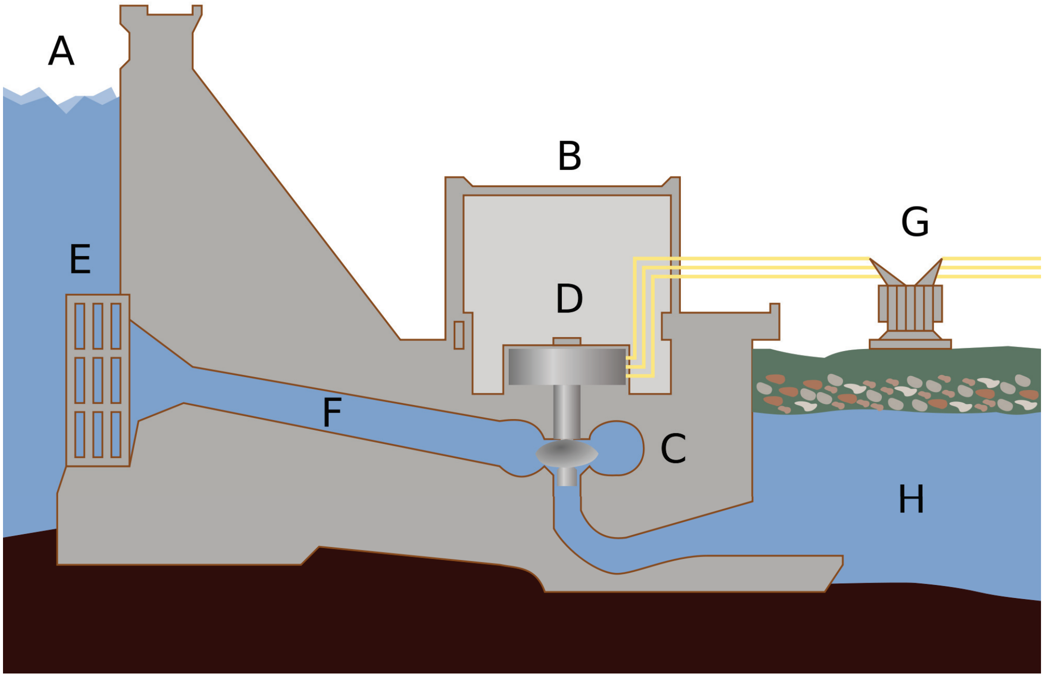

FIGURE 1 – Schéma en coupe d'un barrage hydroélectrique (source [34]). A : réservoir, B : centrale électrique, C : turbine, D : générateur, E : vanne, F : conduite forcée, G : lignes haute tension, H : rivière.

La stabilisation par retour d'état des équations de Navier-Stokes est aussi utilisée pour passer d'un régime turbulent vers un régime laminaire. En effet, dans un circuit (ou système) hydraulique ou oléohydraulique l'écoulement doit toujours être, si possible, laminaire. Au-delà il est en phase dite critique, puis en régime turbulent qui utilise une partie de l'énergie mécanique pour créer des mouvements de plus en plus désordonnés. Les figures 4 et 5 représentent un certain nombre de lignes de courant de l'écoulement

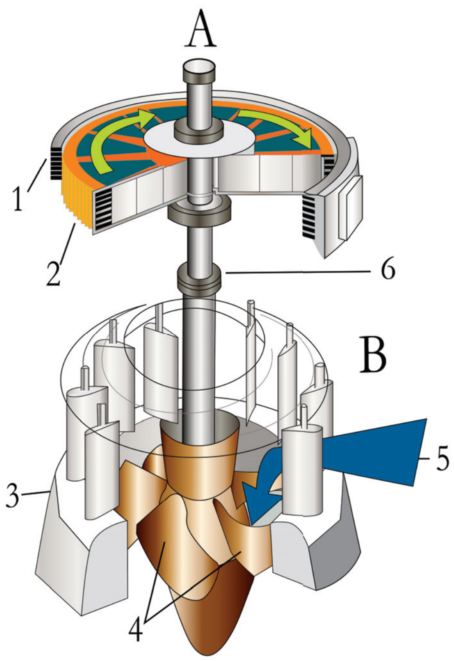

FIGURE 2 – Turbine hydraulique et générateur électrique, vue en coupe (source [35]). A : générateur, 1 : stator , 2 : rotor, B : turbine, 3 : vannes réglables, 4 : pales de la turbine, 5 : flux d'eau, 6 : axe de rotation de la turbine et du générateur.

bidimensionnel dans un domaine borné $\Omega \subset \mathbb{R}^2$, avec un écoulement du type "Poiseuille" en entrée du canal. Sur la Figure 5, en régime turbulent, on observe des tourbillons à l'arrière de l'obstacle cylindrique, connus sous le nom « d'allées de Von Karman ». Lorsqu'un tourbillon se détache, un écoulement dissymétrique se forme autour du corps, ce qui modifie la distribution des pressions. Dans divers problèmes techniques, ce phénomène peut avoir des conséquences dommageables (rupture de ponts suspendus, écroulement de cheminées, accidents d'avion, etc). On s'intéresse alors au problème de stabilisation suivant : comment déterminer une condition limite non homogène, localisée sur la frontière (du cylindre par exemple), permettant de revenir à l'état laminaire ? L'utilisation de parois perforées : méthode d'aspiration-soufflage, permet de mettre en œuvre un contrôle en boucle fermée (aussi appelé contrôle feedback). C'est un contrôle qui dépend à chaque instant de la variable d'état du système et dont la formulation mathématique (9) est donnée après la formulation différentielle du problème de stabilisation (3).

2 Problème de stabilisation

Soit Ω un ouvert connexe borné de classe $\mathcal{C}^2$ dans $\mathbb{R}^d,\ d = 2, 3$, de frontière de $\Gamma = \partial\Omega$. Celle-ci est constituée de deux composantes connexes Γ_l and Γ_b tel que $\Gamma = \Gamma_l \cup \Gamma_b$. En

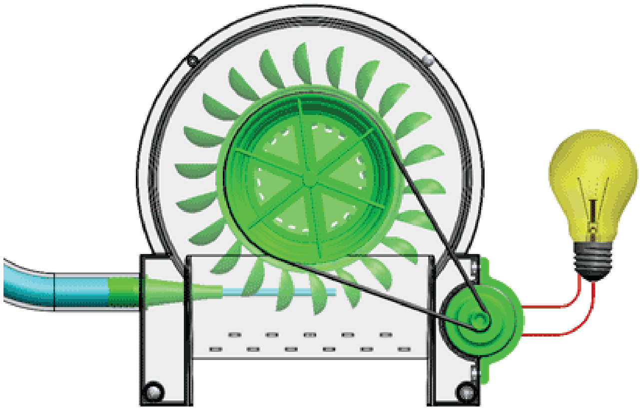

FIGURE 3 – Schéma turbine et générateur électrique (source [36]).

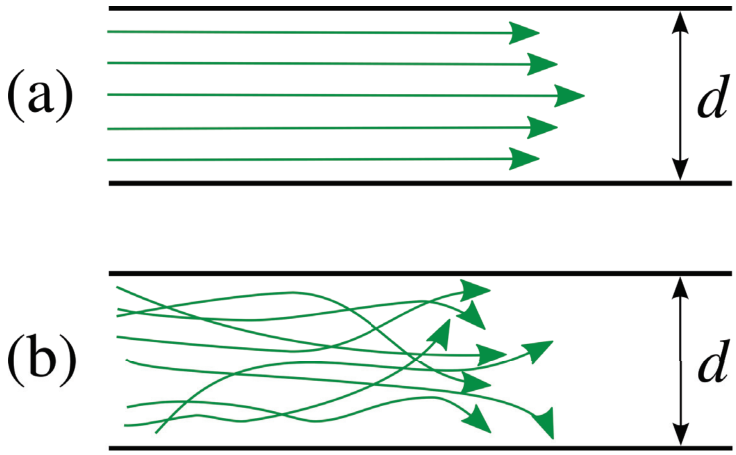

FIGURE 4 – (a) : Écoulement laminaire ; (b) : Écoulement turbulent.

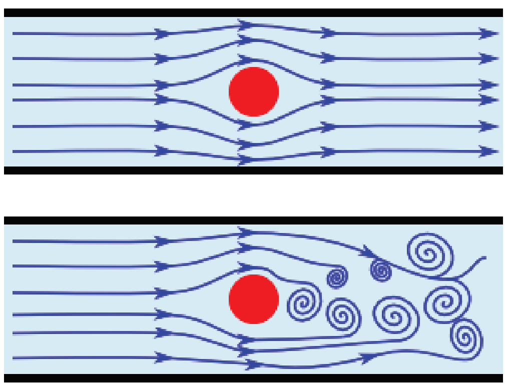

FIGURE 5 – En haut : écoulement laminaire ; En bas : écoulement turbulent.

particulier, le bord Γ_b est la partie de Γ où le contrôle frontière sous forme de feedback est déterminé.

On considère un écoulement incompressible stationnaire dans Ω décrit par les équations de Navier-Stokes adimensionnées [1] suivantes :

$$\begin{cases} (a) & -\dfrac{1}{R_e}\Delta\boldsymbol{\psi} + (\boldsymbol{\psi}\cdot\nabla)\boldsymbol{\psi} + \nabla r = \mathrm{f} & \text{in } \Omega, \\ (b) & \nabla\cdot\boldsymbol{\psi} = 0 & \text{in } \Omega, \\ (c) & \boldsymbol{\psi} = 0 & \text{on } \Gamma_l, \\ (c) & \boldsymbol{\psi} = \boldsymbol{\psi}_b & \text{on } \Gamma_b, \end{cases} \tag{1}$$

où les variables $\boldsymbol{\psi}$, r et les paramètres $\mathrm{f}, \boldsymbol{\psi}_b$ et R_e sont définis comme suit :

- $\boldsymbol{\psi}$: champs de vitesse d'une particule fluide
- r : pression
- f : forces massiques s'exerçant dans le fluide (ex : la gravité)
- $\boldsymbol{\psi}_b$: champs de vitesse au bord
- R_e : nombre de Reynolds (sans dimension)

$$R_e = \frac{U_0 D_0}{\nu}$$

avec

- U_0 - vitesse caractéristique du fluide $[m/s]$
- D_0 - dimension caractéristique $[m]$
- ν - viscosité cinématique du fluide $[m^2/s]$.

Lorsque la force f et le champ de vitesse au bord $\boldsymbol{\psi}_b$ vérifient certaines conditions, l'existence d'une solution $(\boldsymbol{\psi}, r)$ satisfaisant (1) est connue dans [14, 20, 32]. En plus, lorsque le nombre de Reynolds R_e dépasse une certaine valeur critique, le système décrit dans (1) est soumis à une perturbation et le champ de vitesse stationnaire $\boldsymbol{\psi}$ est dit instable.

Supposons maintenant qu'à un instant initial $t = 0$ le champ de vitesse quitte son état d'équilibre $\boldsymbol{\psi}$ et soit égal à $\mathbf{u}(0, \mathbf{x}) \neq \boldsymbol{\psi}(\mathbf{x})$, l'évolution du couple vitesse pression $(\mathbf{u}, q)$ au cours du temps est alors décrite par les équations de Navier-Stokes incompressibles non-

1. Pour faciliter une analyse quantitative des équations de Navier-Stokes, il est d'usage de mettre ces équations sous forme adimensionnée.

stationnaires suivantes :

$$\begin{cases} (a) & \dfrac{\partial \mathbf{u}}{\partial t} - \dfrac{1}{R_e}\Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla q = \mathbf{f}(\mathbf{x}), \quad \nabla\cdot\mathbf{u} = 0 & \text{dans } [0,+\infty[\times\Omega, \\ (b) & \mathbf{u} = 0 & \text{sur } [0,+\infty[\times\Gamma_l, \\ (c) & \mathbf{u} = \mathbf{v}_b + \boldsymbol{\psi}_b & \text{sur } [0,+\infty[\times\Gamma_b, \\ (d) & \mathbf{u}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) + \boldsymbol{\psi}(\mathbf{x}) & \text{dans } \Omega, \end{cases} \tag{2}$$

où $\mathbf{v}_b$ est le contrôle et $\mathbf{v}_0$ la perturbation de l'état d'équilibre.

Problème de stabilisation. En remplaçant $(\mathbf{u}, q)$ par $(\mathbf{v}+\boldsymbol{\psi}, p+r)$ dans (2), puis en utilisant (1), on voit que le coupe $(\mathbf{v}, p)$ satisfait le problème de stabilisation suivant :

$$\begin{cases} (a) & \dfrac{\partial \mathbf{v}}{\partial t} - \dfrac{1}{R_e}\Delta \mathbf{v} + (\mathbf{v}\cdot\nabla)\boldsymbol{\psi} + (\boldsymbol{\psi}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0 & \text{dans } [0,+\infty[\times\Omega, \\ (b) & \nabla\cdot\ \mathbf{v} = 0 & \text{dans } [0,+\infty[\times\Omega, \\ (c) & \mathbf{v} = 0 & \text{sur } [0,+\infty[\times\Gamma_l, \\ (d) & \mathbf{v} = \mathbf{v}_b & \text{sur } [0,+\infty[\times\Gamma_b, \\ (e) & \mathbf{v}(t=0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{sur } \Omega. \end{cases} \tag{3}$$

Forme trilinéaire. Afin de donner l'estimation a priori du problème de stabilisation des équations de Navier-Stokes (3), nous introduisons la forme trilinéaire

$$b(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3) = \int_\Omega (\mathbf{v}_1\nabla)\mathbf{v}_2\cdot\mathbf{v}_3\, d\mathbf{x}, \quad \forall(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)\in \mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega).$$

En intégrant par parties la forme trilinéaire $b(\cdot,\cdot,\cdot)$, on obtient les égalités suivantes :

$$b(\mathbf{u},\mathbf{v},\mathbf{v}) = \frac{1}{2}\int_{\Gamma_b} |\mathbf{v}|^2(\mathbf{u}\cdot\mathbf{n}),\ \forall \mathbf{u},\mathbf{v}\in\mathbf{V}(\Omega), \tag{4}$$

$$b(\mathbf{v},\mathbf{v},\mathbf{v}) = \frac{1}{2}\int_{\Gamma_b} |\mathbf{v}|^2(\mathbf{v}\cdot\mathbf{n}),\ \forall \mathbf{v}\in\mathbf{V}(\Omega), \tag{5}$$

où $\mathbf{V}(\Omega) = \left\{\mathbf{u}\in\mathbf{H}^1(\Omega)\ :\ \nabla\cdot\mathbf{u} = 0,\ \mathbf{u} = 0 \text{ sur } \Gamma_l\right\}$. D'après l'inégalité de Hölder, on a :

$$|b(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)| \le \|\mathbf{v}_1\|\|\nabla\mathbf{v}_2\|_\infty\|\mathbf{v}_3\|, \quad \forall \mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\in\mathbf{H}^1(\Omega), \tag{6}$$

où $\|\cdot\| = \|\cdot\|_{\mathbf{L}^2(\Omega)}$ et $\|\cdot\|_\infty = \|\cdot\|_{\mathbf{L}^\infty(\Omega)}$.

Estimation a priori. Multiplions la première équation de (3) par v et intégrons par partie sur Ω, nous obtenons

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\|\nabla\mathbf{v}\|^2 + b(\mathbf{v},\mathbf{v},\mathbf{v}) + b(\boldsymbol{\psi},\mathbf{v},\mathbf{v}) + b(\mathbf{v},\boldsymbol{\psi},\mathbf{v}) = \int_{\Gamma_b} [\nu\frac{\partial\mathbf{v}}{\partial\mathbf{n}} - p\mathbf{n}]\cdot\mathbf{v}_b. \tag{7}$$

L'utilisation de l'inégalité de Poincaré et des estimations (4)-(6) dans (8) donne

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \beta_\nu\|\mathbf{v}\|^2 \leq \int_{\Gamma_b} [\nu\frac{\partial\mathbf{v}}{\partial\mathbf{n}} - p\mathbf{n}]\cdot\mathbf{v}_b - \frac{1}{2}\int_{\Gamma_b} |\mathbf{v}_b|^2(\boldsymbol{\psi}\cdot\mathbf{n}) - \frac{1}{2}\int_{\Gamma_b} |\mathbf{v}_b|^2(\mathbf{v}_b\cdot\mathbf{n}), \tag{8}$$

où $\beta_\nu = \nu C_p^2 - \|\nabla\boldsymbol{\psi}\|_\infty$ avec C_p la constante de Poincaré.

Notion contrôle feedback. La formulation mathématique de la stabilisation frontière, par contrôle feedback, consiste à trouver $\mathbf{v}_b$ sous la forme

$$\mathbf{v}_b(t) = \mathcal{K}(\mathbf{v}(t)), \quad t \in (0,\infty), \tag{9}$$

où $\mathcal{K}$ est une loi de contrôle à déterminer, de sorte que la vitesse v vérifie par exemple

$$\|\mathbf{v}(t)\|_{\mathbf{X}(\Omega)} \leq C\|\mathbf{v}_0\|_{\mathbf{X}(\Omega)}e^{-\sigma t}, \tag{10}$$

avec $\sigma > 0$ une constante fixée et $\mathbf{X}(\Omega)$ l'espace d'état adéquat. Notons que dans le cadre de cette thèse $\|\cdot\|_{\mathbf{X}(\Omega)} = \|\cdot\|$ avec

$$\mathbf{X}(\Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega) \ : \ \nabla\cdot\mathbf{u} = 0 \text{ dans } \Omega,\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ sur } \Gamma_l\}.$$

Quelques questions.

Selon la réalité que l'on décrit, le problème de stabilisation (3) peut se présenter de plusieurs façons. Par exemple, la fonction ψ dans (3-a) dépend seulement de l'espace mais, en plus de l'espace, elle peut aussi dépendre du temps. En admettant que le bord Γ soit composé de deux parties connexes Γ_0 et Γ_1, des conditions aux limites mixtes (Dirichlet-Neumann) peuvent être considérées aussi. Mais, une fois le problème de stabilisation du type (3) fixé, on pourra se poser quelques questions. Ces questions sont relatives à l'état cible ψ, lequel représente un paramètre important dans un problème de stabilisation.

- Nous dirons que l'état cible (ou état d'équilibre) ψ est stable, dans le sens où $\mathbf{v}_b \equiv 0$ stabilise le problème de stabilisation (3), pour tout $\mathbf{v}_0 \in \mathbf{X}(\Omega)$. Par exemple, $\beta_\nu = \nu C_p^2 - \|\nabla\boldsymbol{\psi}\|_\infty > 0$ dans (8). Cependant, dans le cas où la perturbation initiale v_0 est non nulle sur le bord Γ_b, devons-nous prendre $\mathrm{v}_b \equiv 0$ comme contrôle ? Ou devons construire un contrôle v_b, sous la forme (9), qui stabilise le système (3), progressivement ? De plus, lorsque $\mathrm{v}_0 \neq 0$ sur Γ_b, prendre $\mathrm{v}_b \equiv 0$ entraîne une discontinuité brutale. Avons nous le dispositif (la puissance des vannes, par exemple) permettant d'appliquer ce type de contrôle ? Cette rupture brutale n'entraînera-t-elle pas des conséquences dommageable pour ce dispositif ? Enfin, dans le cas où nous choisissons de contrôler le système de façon progressive, pouvons-nous accélérer la décroissance de l'énergie ?

- Nous dirons que l'état cible (ou état d'équilibre) ψ est instable, dans le sens où, quelque soit $\mathrm{v}_0 \in \mathbf{X}(\Omega)$, $\mathrm{v}_b \equiv 0$ ne stabilise pas le problème (3). Dans ce cas, est-il possible de déterminer une loi de contrôle $\mathcal{K}$ permettant de stabiliser exponentiellement le problème de type (3) ? Notons que lorsqu'un état d'équilibre est instable, une petite perturbation peut entraîner une croissance exponentielle de l'énergie. Étant donné une perturbation initiale v_0 arbitraire dans l'espace fonctionnel $\mathbf{X}(\Omega)$, est-il possible de guider l'état v, initialement en v_0 jusqu'à l'état final $\mathrm{v}_f = 0$? Enfin, puisque le taux de décroissance joue un rôle important dans le processus de stabilisation. On pourra se demander s'il est possible de stabiliser le problème de type (3) pour tout taux de décroissance $\sigma > 0$ fixé i.e.

$$\|\mathbf{v}(t)\|_{\mathbf{V}(\Omega)} \leq C\|\mathbf{v}_0\|_{\mathbf{V}(\Omega)} e^{-\sigma t}, \ t > 0.$$

Nous nous limitons à ces questions même si d'autres interrogations sont possibles. À travers quelques méthodes existantes, nous allons apporter des éléments de réponse à ces questions. Nous commençons cependant par donner les définitions et notations de quelques espaces fonctionnels usuels.

Dans toute la suite, Ω est un ouvert connexe borné de classe $\mathcal{C}^2$ dans $\mathbb{R}^d$, $d = 2, 3$. La frontière de Ω est notée $\Gamma = \partial\Omega$ et elle est constituée de N composantes connexes $\Gamma_1, \ \Gamma_2, \ \Gamma_3, \cdots, \Gamma_N$. On introduit les espaces de fonctions habituels $L^2(\Omega), \ H^s(\Omega), \ H_0^s(\Omega)$ et l'espace dual $H^{-s}(\Omega) = \{H_0^s(\Omega)\}'$. Nous notons en gras les champs de vecteurs $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{H}^s(\Omega) = (H^s(\Omega))^d$, $\mathbf{H}_0^s(\Omega) = (H_0^s(\Omega))^d$ et $\mathbf{H}^{-s}(\Omega) = (H^{-s}(\Omega))^d$. On utilise la notation $\|\cdot\|_{\mathbf{Y}(\Omega)}$ pour les normes, avec en indice l'espace $\mathbf{Y}(\Omega)$ considéré et on note simplement $\langle \cdot \mid \cdot \rangle$ et $\|\cdot\| = \|\cdot\|_{\mathbf{L}^2(\Omega)}$, le produit scalaire et la norme de $\mathbf{L}^2(\Omega)$, respectivement. Les mêmes conventions sont utilisées pour les espaces de traces $L^2(\Gamma)$ et $H^s(\Gamma)$. En plus, si $\mathbf{u} \in \mathbf{L}^2(\Omega)$ est tel que $\nabla \cdot \mathbf{u} \in L^2(\Omega)$, alors nous notons par $\mathbf{u} \cdot \mathbf{n}$ la trace normale de $\mathbf{u}$ dans $\mathbf{H}^{-\frac{1}{2}}(\Gamma)$, où $\mathbf{n}$ est le vecteur unitaire normal de Γ extérieur à Ω. Enfin, dans toute l'introduction, nous notons par $\mathbf{X}(\Omega)$ l'espace de la condition initiale v_0 et par $\mathbf{U}(\Gamma)$ l'espace du contrôle v_b.

3 Méthodes classiques

La question de stabiliser les équations de Navier-Stokes incompressibles avec un contrôle frontière a été étudiée par plusieurs auteurs, e.g. A.V. Fursikov [18, 19], V. Barbu et al. [6, 10, 11, 12, 13], J.-P. Raymond et al. [28, 29, 30] et M. Badra et al. [2, 3, 4]. Dans ces articles, Les auteurs considèrent le problème de stabilisation (3) avec $\psi \equiv \psi(\mathrm{x})$. Ensuite, avec une condition adéquate de Dirichlet au bord, ils transforment le système de stabilisation sous la forme

$$\mathbf{y}' = A\mathbf{y} + B\mathbf{u} + \kappa F(\mathbf{y}, \mathbf{u}), \quad \mathbf{y}(0) = \mathbf{y}_0, \tag{11}$$

où y est la nouvelle variable d'état, u la nouvelle variable de contrôle, A est un opérateur linéaire et est générateur infinitésimal d'un semi-groupe, B est un opérateur linéaire, F une application non-linéaire et $\kappa = 0$ ou 1.

Dans [18, 19], l'auteur construit un opérateur $\mathcal{K}$ à l'aide d'une procédure d'extension de la condition initiale y_0 laquelle nécessite le calcul des vecteurs propres de l'opérateur de Oseen. Il obtient un contrôle de la forme $\mathbf{u} = \mathcal{K}\mathbf{y}_0$ avec

$$\begin{aligned}
\mathbf{X}(\Omega) &= \Big\{\mathbf{u} \in \mathbf{H}^1(\Omega) \ : \ \nabla \cdot \mathbf{u} = 0 \text{ dans } \Omega, \ \mathbf{u} = 0 \ \text{ sur } \Gamma_0, \int_{\Gamma_1} \mathbf{u} \cdot \mathbf{n} = 0\Big\}, \\
\mathbf{U}(\Gamma) &= \Big\{\mathbf{u} \in \mathbf{H}^{3/2}(\Gamma), \ \mathbf{u} = 0 \ \text{ sur } \Gamma_0, \int_{\Gamma_1} \mathbf{u} \cdot \mathbf{n} = 0\Big\},
\end{aligned}$$

où $\Gamma = \Gamma_0 \cup \Gamma_1$ avec $\Gamma_1 \cap \Gamma_0 = \varnothing$. Même si la loi de contrôle $\mathcal{K}$ a été bien caractérisée, elle dépend cependant du temps et de la condition initiale. Notons que les lois de contrôle du type (9) c'est à dire indépendantes du temps et de la condition initiale, sont généralement préférables dans les applications en ingénierie car elles sont plus robustes par rapport aux perturbations dans les modèles. En dimension deux, J. P. Raymond a obtenu dans [30] une loi de contrôle frontière du type (9), où l'opérateur de contrôle $\mathcal{K}$ est déterminé en résolvant une équation algébrique de Riccati obtenue via la solution d'un problème de contrôle optimal. Afin d'obtenir ce résultat cité précédemment, la condition initiale y_0 et le contrôle u doivent respectivement appartenir aux espaces

$$\begin{aligned}
\mathbf{X}(\Omega) &= \Big\{\mathbf{u} \in \mathbf{H}^{1/2-\varepsilon}(\Omega) \ : \ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, \ \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma\Big\}, \\
\mathbf{U}(\Gamma) &= \Big\{m\,\mathbf{u} \in \mathbf{L}^2(\Gamma) : \int_{\Gamma} m\,\mathbf{u} \cdot \mathbf{n}\, d\zeta = 0\Big\},
\end{aligned}$$

où $0 < \varepsilon < 1/4$ et la fonction $m \in C^2(\Gamma)$ à valeurs dans $[0, 1]$ permet de localiser le contrôle u qui n'est appliqué que sur une partie de la frontière. Malheureusement, comme expliqué dans [29], le cas de la dimension trois (3D) est plus exigeant en termes de régularité de la vitesse y et il ne peut pas être traité de la même manière que le cas bidimensionnel. En effet, en 3D le contrôle $\mathbf{u} = \mathcal{K}(\mathbf{y})$ appartient à $\mathbf{H}^{1/4+\varepsilon/2}(0,\infty;\mathbf{L}^2(\Gamma))$ avec $1/2 \leq \varepsilon$, et dans le cas particulier où $1/2 < \varepsilon$, l'espace $\mathbf{H}^{1/4+\varepsilon/2}([0,\infty[;\mathbf{L}^2(\Gamma))$ est un sous espace de $C([0,\infty[;\mathbf{L}^2(\Gamma))$, impliquant ainsi la vitesse initiale à satisfaire la condition de compatibilité au bord $\mathbf{y}_0|_\Gamma = \mathcal{K}(\mathbf{y}_0)$. Plus précisément, pour une donnée initiale $\mathbf{y}_0$ qui ne satisfait pas $\mathbf{y}_0|_\Gamma = \mathcal{K}(\mathbf{y}_0)$, il n'est pas possible d'obtenir une solution avec la méthode de point fixe. Afin de faire coïncider la trace de la condition initiale et le retour d'état frontière initial, l'auteur introduit dans [29] une loi de feedback dépendant du temps sur un intervalle transitoire initial $[0,t_0]$. Cette loi se calcule à l'aide d'une équation de Riccati différentielle sur $[0,t_0[$ et d'une équation de Riccati algébrique sur $[t_0,+\infty[$. En plus, pour obtenir le résultat de stabilisation par l'approche de Riccati, des espaces particuliers de conditions initiales donnés dans [4] sont utilisés.

L'étude réalisée dans [29], améliore d'une certaine façon les résultats obtenus dans [10, 11], où un contrôle frontière tangentiel basé à la fois sur l'approche de Riccati et l'approche spectrale est utilisé. Le cas 3D est très exigeant en termes de régularité de la vitesse. Cependant dans [11], l'auteur établit une loi de feedback du type (9) par la résolution d'un problème de contrôle optimal avec une fonction coût qui met en jeu la norme $L^2(0,\infty;\mathbf{H}^{3/2+\varepsilon}(\Omega))$ de l'état, pour $\epsilon > 0$ assez petit. Par contre, comme expliqué dans [11], l'équation de Riccati dont dépend la loi de feedback est mal posée. Celle-ci est définie faiblement pour un espace de fonctions tests qui dépend de la solution de l'équation. Cette difficulté est intrinsèquement liée à la condition de compatibilité de la trace de la condition initiale. Celle-ci est nécessaire pour obtenir la décroissance exponentielle de la solution des équations de Navier-Stokes en 3D. En effet, pour obtenir cette condition, les auteurs ont choisi un opérateur d'observation trop fortement non borné qui ne permet pas d'obtenir une équation de Riccati en un sens classique. Afin d'obtenir une équation de Riccati bien posée pour $d = 3$, l'auteur choisit dans [29] une fonction coût qui met en jeu une norme très faible de la variable d'état.

Rappelons que dans [29], une loi de feedback dépendant du temps dans un intervalle transitoire initial a été introduite. Comme expliqué dans [3], trouver une loi de contrôle indépendante du temps, laquelle satisfait $\mathbf{y}_0|_\Gamma = \mathcal{K}(\mathbf{y}_0)$ pour une classe donnée de conditions initiales $\mathbf{y}_0$, n'est pas évident. Ce problème est ainsi étudié dans [3] aussi bien en dimension deux qu'en dimension trois, et il a conduit à la recherche du contrôle $\mathbf{v}_b$ dans un système étendu composé du problème d'évolution

$$\frac{\partial \mathbf{v}_b}{\partial t} - \Delta_B \mathbf{v}_b - \sigma\,\mathbf{n} = \mathcal{K}(\mathbf{v},\mathbf{v}_b), \quad \mathbf{v}_b(0) = \mathbf{v}_0|_\Gamma,$$

couplé avec le système de Navier-Stokes original, où Δ_B représente l'opérateur de Laplace-Beltrami et la loi de contrôle $\mathcal{K}$ agit maintenant sur le couple $(\mathbf{v},\mathbf{v}_b)$. Dans cette étude, l'espace fonctionnel $\mathbf{X}(\Omega)$ de la condition initiale est défini comme suit

$$\mathbf{X}(\Omega) = \left\{\mathbf{u} \in \mathbf{H}^s(\Omega) \ : \ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, \ \int_\Gamma \mathbf{u} \cdot \mathbf{n} = 0\right\},$$

avec $s \in [\frac{d-2}{2}, 1] \backslash \{1/2\}$, l'opérateur $\mathcal{K}$ est obtenu à partir d'une équation de Riccati bien posé et le contrôle v_b est défini sur une partie Γ arbitrairement choisie.

Afin de stabiliser les équations de Navier-Stokes autour d'un état stationnaire, sauf dans les papiers de A.V. Fursikov [18, 19], des lois de feedback sont déterminées en résolvant une équation de Riccati algébrique de dimension infinie [3, 4, 9, 10, 11, 29, 30]. Bien que notre étude ne porte que sur la construction d'un contrôle frontière, l'approche de Riccati de dimension infinie évoquée ci-dessus, s'applique aussi aux cas de contrôles internes [7, 13]. Dans le cas où le contrôle est obtenue en résolvant une équation algébrique de Riccati sur un espace de dimension infinie, un problème de contrôle optimal est résolu. Dans la pratique, ce problème est très difficile à mettre en œuvre. En effet, les matrices apparaissant dans la version discrète de l'équation de Riccati sont de très grande dimension, et la solution Π de cette équation tel que $\mathcal{K} = -B^*\Pi$ par exemple, est également une matrice de grande taille et pleine. Cela entraîne des problèmes de mémoire rendant la résolution numérique difficile. En conséquence, l'utilisation de contrôleurs de dimension finie peut être plus appropriée pour stabiliser les équations de Navier-Stokes. Notons qu'une telle approche est étudiée dans [8, 12], dans le cas d'un contrôle interne, et dans [2, 6, 28], dans le cas d'un contrôle frontière. Dans ces études citées ci-dessus, les auteurs cherchent un contrôle frontière u de dimension finie de la forme

$$\mathbf{u} = \sum_{j=1}^{N} u_j(t)\boldsymbol{\varphi}_j(\mathbf{x}), \qquad t \geq 0, \ \mathbf{x} \in \Gamma, \tag{12}$$

où

- N : est la taille de l'espace instable d'un certain opérateur $\mathcal{A}$, c'est à dire si $(\lambda_k)_{k\in\mathbb{N}^*}$ représente l'ensemble des valeurs propres complexes de $\mathcal{A}$, N est tel que, pour tout taux de décroissance $\sigma > 0$ fixé

$$\cdots \leq \Re\,\lambda_{N+1} < -\sigma < \Re\,\lambda_N \leq \cdots \leq \Re\,\lambda_2 \leq \Re\,\lambda_1. \tag{13}$$

- $\boldsymbol{\varphi}_j$, $j = 1, 2, 3, \ldots, N$: est la fonction propre de $\mathcal{A}$ associée à la valeur propre λ_j.

- u_j, $j = 1, 2, 3, \ldots, N$: est exprimé sous forme de feedback.

Dans [28] où le cas 2D est traité, un contrôle v_b de la forme (12) est obtenu à partir de la solution d'une équation de Riccati de dimension finie dans $\mathbb{R}^{n_c \times n_c}$, où n_c est la taille de l'espace instable de l'opérateur d'Oseen. La même approche est ensuite élargie dans [2] pour le cas de la dimension trois. En contrôle interne dans [8], au lieu d'utiliser l'approche de Riccati, une technique de stabilisation stochastique est utilisée. Celle-ci permet d'éviter les difficultés liées à la dimension infinie des équations de Riccati. Ensuite, une procédure semblable est utilisée dans [6] dans le cadre d'un contrôle frontière.

Dans toutes les études mentionnées ci-dessus, une loi de feedback linéaire est d'abord déterminée en résolvant un problème de contrôle linéaire ($\kappa = 0$ dans (11)). Ensuite cette loi de contrôle linéaire est utilisée pour stabiliser le système non linéaire. Un tel procédé impose de choisir une vitesse initiale assez petite. En plus, les méthodes employées (par exemple, l'approche de Riccati) exigent de chercher la condition initiale $\mathbf{y}_0$ dans des espaces suffisamment réguliers, selon que $d = 2$ ou $d = 3$. Par exemple, dans [6, Theorem 2.3] pour $d = 2$ on a $\mathbf{y}_0 \in \mathbf{X}(\Omega) = \mathbf{H}^{1/2-\epsilon}(\Omega) \cap \widetilde{\mathbf{H}}(\Omega)$ où

$$\widetilde{\mathbf{H}}(\Omega) \quad = \quad \left\{\mathbf{u} \in \mathbf{L}^2(\Omega) \ : \nabla \cdot \mathbf{u} = 0, \ \ \mathbf{u} \cdot \mathbf{n} = 0 \ \text{ sur } \Gamma\right\}, \tag{14}$$

tandis que dans [2, Theorem 2], pour $d = 3$, on a $\mathbf{y}_0 \in \mathbf{H}_0^s(\Omega)$, $s \in (1/2, 1]$ avec $\nabla \cdot \mathbf{y}_0 = 0$. Nous avons aussi vu que le taux de décroissance σ est arbitrairement choisi dans la plus part de ces études citées, par exemple [2, 6, 18, 19]. Une fois fixé, ce taux détermine la valeur de $C \geq 1$ dans (10) et la taille de N dans (13). Cependant, dans ces publications les auteurs ne précisent pas les valeurs exactes de ces deux constantes. Notons que les valeurs propres $(\lambda_k)_{k\in\mathbb{N}^*}$ de $\mathcal{A}$ dépendent du nombre de Reynolds R_e (ou de la viscosité ν), de l'état stationnaire ψ et du domaine Ω. Par conséquent, même pour un taux σ petit, la taille de N peut être très grande car dépendant de la répartition des $(\Re\lambda_k)_{k\in\mathbb{N}^*}$ dans $\mathbb{R}$. En ce qui concerne le C dans (10), notons que travailler avec une constante très grande n'est pas souhaitable car ce phénomène pourrait entraîner une croissance exponentielle de l'énergie au début du processus.

Il existe cependant des méthodes qui ne cherchent pas une loi de feedback par la résolution d'un problème de contrôle linéaire. Dans [5] avec une approche différente, les auteurs étudient le problème de la stabilisation par contrôle frontière des équations de Navier-Stokes 2D dans un canal borné. Leur approche consiste à trouver une loi de feedback en utilisant un actionnement de la vitesse tangentielle. Cette loi a permis aux auteurs d'obtenir un résultat de stabilité du type (10), avec C=1, pour une vitesse initiale arbitrairement choisie dans l'espace fonctionnel $\widetilde{\mathbf{H}}(\Omega)$ défini dans (14).

Nous allons maintenant introduire la méthode utilisée dans cette thèse et présenter ses avantages.

4 Nouvelle méthode

Dans cette thèse au lieu de chercher une loi de feedback par la résolution d'un problème de contrôle linéaire, éventuellement par la résolution d'une équation de Riccati, une nouvelle approche est proposée. Celle-ci diffère aussi de l'approche proposée dans

[5]. Elle consiste à établir une équation impliquant la dérivée de l'énergie par rapport au temps et les conditions aux limites. La décroissance exponentielle de l'énergie est obtenue en choisissant des conditions aux limites adéquates. Cette méthode a été développée pour la première fois dans [31] pour la stabilisation du système de Saint-Venant 1D, ensuite elle a été appliquée dans [22, 23, 17]. Dans [22, 23], avec les équations de Saint-Venant, les auteurs stabilisent les réseaux de canaux d'irrigation, tandis que dans [17] les auteurs traitent un système couplant les équations de Saint-Venant aux équations érosion-sédimentation. Cette thèse est cependant consacrée à l'étude du problème de stabilisation par retour d'état ou "feedback" des équations de Navier-Stokes incompressibles autour d'un état stationnaire ou d'un état non-stationnaire donné. Bien que certains auteurs utilisent le contrôle interne (contrôle effectué sur une partie interne du domaine) pour stabiliser le problème de Navier-Stokes incompressible, le cas d'un contrôle localisé sur la frontière du domaine est considéré dans cette thèse. Le contrôle s'exprime en fonction du champ de vitesse à l'aide d'une loi de feedback non-linéaire. Celle-ci est fournie grâce aux techniques d'estimation a priori via la méthode Faedo-Galerkin laquelle consiste à construire une suite de solutions approchées en utilisant une base de Galerkin adéquate. Cette loi de feedback assure la décroissance exponentielle de l'énergie du problème discret correspondant. Le système satisfait par les solutions approchées étant non-linéaire, le passage à la limite se fait grâce au résultat de compacité [26].

L'approche proposée dans cette thèse présente plusieurs avantages. Elle permet d'étudier la stabilisation exponentielle (par contrôle frontière) des équations Navier-Stokes non seulement autour d'un état stationnaire, mais aussi autour d'un état non-stationnaire. La méthode permet aussi de stabiliser le problème de Navier-Stokes avec des conditions aux bords mixtes (Dirichlet-Neumann) autour d'un état d'équilibre donné. À notre connaissance, l'étude théorique de la stabilisation exponentielle par contrôle frontière des équations de Navier-Stokes autour d'un état non-stationnaire et la stabilisation exponentielle par contrôle frontière des équations de Navier-Stokes avec des conditions aux bords mixtes autour d'un état d'équilibre donné n'a pas été abordée dans la littérature. En plus, le résultat de stabilisation $\|\mathbf{v}(t,\mathbf{x})\| \leq \|\mathbf{v}_0(\mathbf{x})\| e^{-\sigma t}$, $t \in (0,\infty)$, est obtenu pour un certain $\sigma > 0$ et pour une vitesse initiale $\mathbf{v}_0$ arbitrairement choisie dans $\mathbf{H}(\Omega) = \left\{\mathbf{u} \in \mathbf{L}^2(\Omega)\ :\nabla\cdot\mathbf{u} = 0,\ \ \mathbf{u}\cdot\mathbf{n} = 0 \ \text{ sur } \Gamma_l\right\}$. Cet espace impose moins de régularité à $\mathbf{v}_0$ comparé aux résultats cités ci-dessus, par exemple voir [6, Theorem 2.3] et ce résultat de régularité est indépendant de la dimension $d = 2$ ou $d = 3$.

Dans la suite de cette introduction, nous présentons de manière plus détaillée le contenu de chacun des chapitres de cette thèse.

5 Description des résultats obtenus

5.1 Stabilisation frontière des équations de Navier-Stokes par un contrôle feedback via une méthode de Galerkin

On considère un domaine ouvert Ω de $\mathbb{R}^d$ ($d = 2$ ou $d = 3$), borné connexe de classe C^2 et de frontière $\partial\Omega = \Gamma$. Celle-ci est constituée de deux composantes connexes Γ_l and Γ_b tel que $\Gamma = \Gamma_l \cup \Gamma_b$. En particulier, le bord Γ_b est la partie de Γ où le contrôle frontière sous forme de feedback est déterminé. On considère dans Ω, un écoulement incompressible stationnaire décrit par le couple $(\mathbf{v}_s, q_s)$, solution système de Navier-Stokes suivant

$$\begin{cases} -\nu\Delta\mathbf{v}_s + (\mathbf{v}_s.\nabla)\mathbf{v}_s + \nabla q_s = \mathbf{f}_s & \text{dans } \Omega, \\ \nabla\cdot\mathbf{v}_s = 0 & \text{dans } \Omega, \\ \mathbf{v}_s = \mathbf{v}_b & \text{sur } \Gamma_b, \\ \mathbf{v}_s = 0 & \text{sur } \Gamma_l, \end{cases} \tag{15}$$

où la viscosité ν est strictement positive, le champ de force $\mathbf{f}_s$ est dans $\mathbf{H}^{-1}(\Omega)$ et la condition au bord $\mathbf{v}_b$ appartient à

$$V^{1/2}(\Gamma_b) = \Big\{\mathbf{u} \in \mathbf{H}^{1/2}(\Gamma_b) : \int_{\Gamma_b} \mathbf{u}\cdot\mathbf{n}\, d\zeta = 0\Big\}.$$

Rappelons qu'une solution $(\mathbf{v}_s, q_s)$ de (15), appartenant à $\mathbf{H}^1(\Omega) \times L^2_0(\Omega)$, est connue dans [20], où $L^2_0(\Omega)$ est l'espace des pressions à valeur moyenne nulle :

$$L^2_0(\Omega) = \left\{p \in L^2(\Omega),\ \int_\Omega p(\mathbf{x})\, d\mathbf{x} = 0\right\}.$$

Soit $T > 0$ un réel fixé, on pose

$$Q = [0, T[\times\Omega, \quad \Sigma_l = [0, T[\times\Gamma_l \quad \text{et} \quad \Sigma_b = [0, T[\times\Gamma_b$$

et on considère le problème de Navier-Stokes non-stationnaire suivant

$$\begin{cases} \dfrac{\partial\mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla q = \mathbf{f}_s & \text{dans } Q, \\ \nabla\cdot\mathbf{u} = 0 & \text{dans } Q, \\ \mathbf{u} = \mathbf{v}_b + \mathbf{u}_b & \text{sur } \Sigma_b, \\ \mathbf{u} = 0 & \text{sur } \Sigma_l, \\ \mathbf{u}_0(\mathbf{x}) = \mathbf{v}_s(\mathbf{x}) + \mathbf{v}_0(\mathbf{x}) & \text{dans } \Omega. \end{cases} \tag{16}$$

Le couple $(\mathbf{v} = \mathbf{u} - \mathbf{v}_s, p = q - q_s)$ satisfait alors le problème suivant

$$\begin{cases} (a) & \dfrac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}_s + (\mathbf{v}_s\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0 & \text{dans } Q, \\ (b) & \nabla\cdot\ \mathbf{v} = 0 & \text{dans } Q, \\ (c) & \mathbf{v} = \mathbf{u}_b & \text{sur } \Sigma_b, \\ (d) & \mathbf{v} = 0 & \text{sur } \Sigma_l, \\ (e) & \mathbf{v}(t=0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{dans } \Omega. \end{cases} \tag{17}$$

Le but du chapitre 1 est de trouver, via le système (17), un contrôle $\mathbf{u}_b$ sur Σ_b qui permet de stabiliser le problème de Navier-Stokes (16) autour de l'état stationnaire $\mathbf{v}_s$.

Nous résumons les parties essentielles de ce chapitre, ensuite énonçons le résultat de stabilisation obtenu.

Espaces fonctionnels. On considère les espaces des fonctions à divergence nulle

$$\mathbf{V}(\Omega) = \Big\{\mathbf{u} \in \mathbf{H}^1(\Omega) \ :\ \nabla\cdot\mathbf{u} = 0 \text{ dans } \Omega,\ \mathbf{u} = 0 \text{ sur } \Gamma_l, \int_{\Gamma_b} \mathbf{u}\cdot\mathbf{n}\, d\zeta = 0\Big\}, \tag{18}$$

$$\mathbf{V}_0(\Omega) = \{\mathbf{u} \in \mathbf{H}^1_0(\Omega) \ :\ \nabla\cdot\mathbf{u} = 0 \text{ dans } \Omega\}, \tag{19}$$

$$\mathbf{H}(\Omega) = \Big\{\mathbf{u} \in \mathbf{L}^2(\Omega) \ :\nabla\cdot\mathbf{u} = 0,\ \ \mathbf{u}\cdot\mathbf{n} = 0 \ \text{ sur } \Gamma_l, \int_{\Gamma_b} \mathbf{u}\cdot\mathbf{n}\, d\zeta = 0\Big\}. \tag{20}$$

Nous avons, par définition $\|\cdot\|_{\mathbf{V}(\Omega)} = \|\cdot\|_{\mathbf{H}^1(\Omega)}$, car $\mathbf{V}(\Omega)$ est un sous espace fermé de $\mathbf{H}^1(\Omega)$.

Définition 5.1. *On désigne par $\mathbf{V}^{1/2}(\Gamma_b)$ le sous-espace de $\mathbf{H}^{1/2}(\Gamma)$ formé des fonctions définies dans Γ_b et dont l'extension par zéro sur $\Gamma\setminus\Gamma_b$ appartient à $\mathbf{H}^{1/2}(\Gamma)$. Soit $\mathbf{g} \in \mathbf{V}^{1/2}(\Gamma_b)$ tel que $\mathbf{g}\cdot\mathbf{n} \neq 0$ sur Γ_b et $\int_{\Gamma_b} \mathbf{g}\cdot\mathbf{n}\, d\zeta = 0$, on définit par*

$$W(Q) = \{(\mathbf{v},\alpha) \in \mathbf{V}(\Omega)\times\mathbb{R},\ \textit{tel que } \mathbf{v} = \alpha\mathbf{g} \textit{ sur } \Gamma_b\}, \tag{21}$$

l'espace fonctionnel dans lequel la solution $\mathbf{v}$ de (17) sera cherchée.

Formes Linéaires. Afin de définir la formulation faible du problème de stabilisation des équations de Navier-Stokes, nous introduisons la forme bilinéaire

$$a(\mathbf{v}_1,\mathbf{v}_2) = \int_\Omega \nabla\mathbf{v}_1 : \nabla\mathbf{v}_2\, d\mathbf{x}, \quad \forall(\mathbf{v}_1,\mathbf{v}_2) \in \mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega),$$

et la forme trilinéaire

$$b(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3) = \int_\Omega (\mathbf{v}_1\nabla)\mathbf{v}_2\cdot\mathbf{v}_3\, d\mathbf{x}, \quad \forall(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3) \in \mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega).$$

En intégrant par parties la forme trilinéaire $b(\cdot,\cdot,\cdot)$, on obtient les propriétés suivantes :

$$b(\mathbf{u},\mathbf{v},\mathbf{v}) = \frac{\alpha^2}{2}\int_{\Gamma_b} |\mathbf{g}|^2(\mathbf{u}\cdot\mathbf{n})\,d\zeta,\ \forall\mathbf{u}\in\mathbf{V}(\Omega),\ \forall(\mathbf{v},\alpha)\in W(Q), \tag{22}$$

$$b(\mathbf{v},\mathbf{v},\mathbf{v}) = \frac{\alpha^3}{2}\int_{\Gamma_b} |\mathbf{g}|^2(\mathbf{g}\cdot\mathbf{n})\,d\zeta,\ \forall(\mathbf{v},\alpha)\in W(Q). \tag{23}$$

En plus, d'après l'inégalité de Hölder, on a :

$$|b(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)| \leq \|\mathbf{v}_1\|\|\nabla\mathbf{v}_2\|_\infty\|\mathbf{v}_3\|, \quad \forall\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\in\mathbf{H}^1(\Omega), \tag{24}$$

où $\|\cdot\|_\infty = \|\cdot\|_{\mathbf{L}^\infty(\Omega)}$.

Nous allons maintenant construire une base de Galerkin pour l'espace $W(Q)$.

Base de Galerkin pour $W(Q)$. Soient $\{\mathbf{z}_j,\lambda_j, j=1,2,3,\cdots\}$ les fonctions propres et les valeurs propres du problème spectral de l'opérateur de Stokes suivant :

$$-\Delta\mathbf{z}_j+\nabla p_j=\lambda_j\mathbf{z}_j, \qquad \nabla\cdot\mathbf{z}_j=0 \quad \text{in } \Omega; \qquad \mathbf{z}_j|_\Gamma=0. \tag{25}$$

Comme montré dans [32], $0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_j\to\infty$ lorsque $j\to\infty$. En plus, les $\{\mathbf{z}_j\}$ forment une base orthonormale dans $\mathbf{V}_0(\Omega)$ vérifiant :

$$\begin{cases} \langle\mathbf{z}_j,\mathbf{z}_k\rangle=\delta_{jk}, \\ \langle\nabla\mathbf{z}_j,\nabla\mathbf{z}_k\rangle=\lambda_j\delta_{jk}, \quad \forall j,k=1,2,3,... \end{cases} \tag{26}$$

L'espace $W(Q)$, défini dans (21), est alors réécrit comme suit :

$$W(Q)=\text{span}(\mathbf{z}_n)_{\{n\in\mathbb{N}^*\}}\oplus\text{span}(\mathbf{w}), \tag{27}$$

où w satisfait le système suivant :

$$-\Delta\mathbf{w}+\nabla q=0, \quad \nabla\cdot\mathbf{w}=0 \text{ in } \Omega, \quad \mathbf{w}=0 \text{ on } \Gamma_l, \quad \mathbf{w}=\mathbf{g} \text{ on } \Gamma_b. \tag{28}$$

Puisque g satisfait $\int_{\Gamma_b}\mathbf{g}\cdot\mathbf{n}\,d\zeta=0$, le système (28) admet alors une unique solution $(\mathbf{w},q)$ dans $\mathbf{V}(\Omega)\times L_0^2(\Omega)$ (voir [32]).

Problème de stabilisation. Pour stabiliser le système (17), nous choisissons de chercher la solution v sous la forme $\mathbf{v}=\mathbf{z}+\alpha\mathbf{w}$, où $\mathbf{z}\in\mathbf{V}_0(\Omega)$, w vérifie (28) et α, grâce aux techniques d'estimation a priori, satisfait :

$$\int_{\Gamma_b}[\nu\frac{\partial\mathbf{v}}{\partial\mathbf{n}}-p\mathbf{n}]\cdot\mathbf{g}\,d\zeta=f(\mathbf{v},\alpha), \tag{29}$$

avec

$$f(\mathbf{v},\alpha) = a\alpha^2 + b\alpha - \sigma_0\|\mathbf{v}\|^2\alpha - \nu\lambda_1\left(\|\mathbf{w}\|^2\alpha + 2\langle \mathbf{w},\mathbf{z}\rangle\right). \tag{30}$$

Notons que dans (30)

$$a = \frac{1}{2}\int_{\Gamma_b} |\mathbf{g}|^2(\mathbf{g}\cdot\mathbf{n})\, d\zeta \quad \text{et} \quad b = \frac{1}{2}\int_{\Gamma_b} |\mathbf{g}|^2(\mathbf{v}_s\cdot\mathbf{n})\, d\zeta,$$

λ_1 est la plus petite valeur propre de (25), σ_0 est une constante positive arbitrairement choisie et a été introduite pour limiter la taille du contrôle.
Puisque $\mathbf{v} = \mathbf{z} + \alpha\mathbf{w}$, on a $\mathbf{v} = \alpha\mathbf{g}$ sur Γ_b car $\mathbf{z} = 0$ sur Γ. En couplant le système (17) avec l'équation (29), le couple $(\mathbf{v}, p)$ satisfait maintenant le problème de stabilisation suivant :

$$\begin{cases} (a) & \dfrac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}_s + (\mathbf{v}_s\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0 & \text{dans } Q, \\ (b) & \nabla\cdot\ \mathbf{v} = 0 & \text{dans } Q, \\ (c) & \mathbf{v} = \alpha(t)\mathbf{g}(\mathbf{x}) & \text{sur } \Sigma_b, \\ (d) & \mathbf{v} = 0 & \text{sur } \Sigma_l, \\ (e) & \mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{dans } \Omega, \\ (f) & \displaystyle\int_{\Gamma_b} [\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}]\cdot\mathbf{g}\, d\zeta = f(\mathbf{v},\alpha). \end{cases} \tag{31}$$

Le contrôle α est a priori inconnu et grâce à l'équation (31-f), il satisfait une loi de feedback non linéaire conduisant à chercher des $\alpha(\mathbf{v})$. Puisque (31-f) est indépendant de x, la fonction $\alpha(\mathbf{v})$ dépend uniquement du temps. Pour simplifier, $\alpha(\mathbf{v})$ est noté α dans la suite.

Formulation variationnelle. Nous considérons la formulation variationnelle du problème de stabilisation (31).

Définition 5.2. *Soit $T > 0$ un nombre réel arbitraire, nous dirons que $(\mathbf{v},\alpha)$ est solution faible de (31) sur $[0,T)$ si*

- $\mathbf{v} \in [L^\infty(0,T;\mathbf{H}(\Omega)) \cap L^2(0,T;\mathbf{V}(\Omega))]$,
- $\exists\alpha \in L^\infty(0,T)$ *tel que* $\mathbf{v} = \alpha\mathbf{g}$ *sur* Γ_b,

$$\begin{cases} (a) & \langle d_t\mathbf{v}, \widetilde{\mathbf{v}}\rangle + \nu a(\mathbf{v},\widetilde{\mathbf{v}}) + b(\mathbf{v},\mathbf{v}_s,\widetilde{\mathbf{v}}) + b(\mathbf{v}_s,\mathbf{v},\widetilde{\mathbf{v}}) + b(\mathbf{v},\mathbf{v},\widetilde{\mathbf{v}}) = \widetilde{\alpha} f(\mathbf{v},\alpha), \\ (b) & \mathbf{v}(0) = \mathbf{v}_0, \end{cases} \tag{32}$$

pour tout $(\widetilde{\mathbf{v}},\widetilde{\alpha}) \in W(Q)$.

Résultat de stabilité. Dans le chapitre 1 nous prouvons le théorème suivant :

Théorème 5.3. *Soit λ_1 la plus petite valeur propre de (25). Supposons que l'état stationnaire $\mathbf{v}_s$, la vitesse initiale $\mathbf{v}_0$ et le profil $\mathbf{g}$ satisfont respectivement*

$$\bar{\sigma} = \nu\lambda_1 - \|\nabla \mathbf{v}_s\|_\infty > 0, \tag{33}$$

$$\mathbf{v}_0 \in \mathbf{H}(\Omega), \quad (\mathbf{v}_0 \cdot \mathbf{n})\,\mathbf{n} \in \mathbf{H}^{1/2}(\Gamma_b), \tag{34}$$

$$\mathbf{g} \in \mathbf{V}^{1/2}(\Gamma_b) \quad \textit{and} \quad \alpha_0\,\mathbf{g}\cdot\mathbf{n} = \mathbf{v}_0\cdot\mathbf{n} \;\textit{ on }\; \Gamma_b \textit{ with } \;\mathbf{g}\cdot\mathbf{n} \neq 0,\, \alpha_0 \in \mathbb{R}. \tag{35}$$

Pour toute condition initiale $\mathbf{v}_0$, arbitraire et satisfaisant (34), il existe une solution $(\mathbf{v}, \alpha)$ dans le sens de la définition 5.2, et une distribution p sur Ω tel que (31) soit vérifié. En plus, $\mathbf{v}$ satisfait les estimations suivantes :

$$\|\mathbf{v}(t)\| \;\leq\; \|\mathbf{v}_0\|\, e^{-\sigma(t)},\; \forall t > 0, \tag{36}$$

$$\int_0^T \|\nabla \mathbf{v}(t)\|^2 dt \;\leq\; C\|\mathbf{v}_0\|^2, \tag{37}$$

où la constante $C > 0$, $\sigma(t) = \sigma_1 t + \sigma_0 \int_0^t \alpha^2(s)ds \geq 0$ avec $\sigma_0 > 0$ et $0 < \sigma_1 \leq \bar{\sigma}$.

Remarque 5.4. *Le taux de décroissance $\sigma(t)$ est fonction du contrôle α.*

Remarque 5.5. *Avec la condition (33), la cible $\mathbf{v}_s$ est naturellement stable dans le sens où, si α est identiquement nul ($\alpha \equiv 0$), le système (31) se stabilise seul. Cependant, si la condition initiale $\mathbf{v}_0$ et le profil $\mathbf{g}$ sont tels que $\alpha_0\mathbf{g}\cdot\mathbf{n} = \mathbf{v}_0\cdot\mathbf{n} \neq 0$ sur Γ_b, par exemple, le contrôle α n'est pas identiquement nul (voir Proposition 3.1).*

5.2 Stabilisation frontière du modèle de Navier-Stokes par contrôle feedback autour d'un état non-stationnaire

On considère ici un domaine ouvert Ω de $\mathbb{R}^d$ ($d = 2$ ou $d = 3$), borné connexe de classe C^2 et de frontière Γ. Celle-ci est constituée de deux composantes connexes Γ_l et Γ_b tel que $\Gamma = \Gamma_l \cup \Gamma_b$. En particulier, le bord Γ_b est la partie de Γ, où le contrôle frontière sous forme de feedback est déterminé. Soit $T > 0$ un nombre réel fixé, on pose $Q = [0, T[\times\Omega$, $\Sigma_l = [0, T[\times\Gamma_l$, $\Sigma_b = [0, T[\times\Gamma_b$ et on considère le couple $(\boldsymbol{\psi}, q)$ solution du système de Navier-Stokes non-stationnaire suivant

$$\begin{cases} \dfrac{\partial \boldsymbol{\psi}}{\partial t} - \nu\Delta\boldsymbol{\psi} + (\boldsymbol{\psi}\cdot\nabla)\boldsymbol{\psi} + \nabla q = \mathbf{f} & \text{dans } \; Q, \\ \nabla\cdot\boldsymbol{\psi} = 0 & \text{dans } \; Q, \\ \boldsymbol{\psi} = 0 & \text{sur } \; \Sigma_l, \\ \boldsymbol{\psi} = \boldsymbol{\psi}_b & \text{sur } \; \Sigma_b, \end{cases} \tag{38}$$

où $\nu > 0$ est la viscosité du fluide, f représente la force interne agissant sur le fluide et ψ_b la condition au bord sur Γ_b. On dira qu'une solution $\psi(t,\mathbf{x})$ de (38) appartient à l'ensemble des vitesses admissibles $\mathcal{U}_{ad}$ si elle vérifie

$$\sup_{t\leq T}\|\nabla\boldsymbol{\psi}(t,\mathbf{x})\| < \frac{\nu}{C_\Omega}, \tag{39}$$

où $\|\cdot\| = \|\cdot\|_{(L^2(\Omega))^d}$ et C_Ω est une constante positive définie plus tard dans (46).

On considère une trajectoire $(\mathbf{u}, p)$, solution des équations de Navier-Stokes non-stationnaires

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla r = \mathbf{f} & \text{dans } Q,\\ \nabla\cdot\mathbf{u} = 0 & \text{dans } Q,\\ \mathbf{u} = 0 & \text{sur } \Sigma_l,\\ \mathbf{u} = \mathbf{v}_b + \boldsymbol{\psi}_b & \text{sur } \Sigma_b,\\ \mathbf{u}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) + \boldsymbol{\psi}(0,\mathbf{x}) & \text{dans } \Omega. \end{cases} \tag{40}$$

où $\mathbf{v}_b$ représente le contrôle et $\mathbf{v}_0$ peut être considéré comme une perturbation de l'état initial (38). En remplaçant $(\mathbf{u}, r) = (\mathbf{v}+\boldsymbol{\psi}, p+q)$ dans (40), on obtient le système suivant

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\boldsymbol{\psi} + (\boldsymbol{\psi}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0 & \text{dans } Q,\\ \nabla\cdot\ \mathbf{v} = 0 & \text{dans } Q,\\ \mathbf{v} = \mathbf{v}_b & \text{sur } \Sigma_b,\\ \mathbf{v} = 0 & \text{sur } \Sigma_l,\\ \mathbf{v}(t=0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{dans } \Omega. \end{cases} \tag{41}$$

L'objectif du chapitre 2 est de stabiliser, via le système (41), le problème de Navier-Stokes (40) autour d'un état non-stationnaire $\psi \in \mathcal{U}_{ad}$.

Nous allons maintenant résumer les différentes parties de ce chapitre et énoncer le résultat principal.

Définition 5.6. *On désigne par $\mathbf{V}^{1/2}(\Gamma_b)$ le sous-espace de $\mathbf{H}^{1/2}(\Gamma)$ formé des fonctions définies dans Γ_b et dont l'extension par zéros sur $\Gamma\setminus\Gamma_b$ appartient à $\mathbf{H}^{1/2}(\Gamma)$. Soit $\mathbf{g}\in\mathbf{V}^{1/2}(\Gamma_b)$ tel que $\mathbf{g}\cdot\mathbf{n}\neq 0$ sur Γ_b et $\int_{\Gamma_b}\mathbf{g}\cdot\mathbf{n}\,d\zeta = 0$, on définit l'espace fonctionnel $W(Q)$ par*

$$W(Q) = \{(\mathbf{v},\alpha)\in\mathbf{V}(\Omega)\times\mathbb{R}\ \textit{tel que}\ \mathbf{v} = \alpha\mathbf{g}\ \textit{sur}\ \Gamma_b\}. \tag{42}$$

Notons que la solution de (41) est cherchée dans l'espace fonctionnel $W(Q)$.

Formes Linéaires. Dans le but de définir la formulation faible du problème de stabilisation des équations de Navier-Stokes, nous introduisons la forme bilinéaire

$$a(\mathbf{v}_1, \mathbf{v}_2) = \int_\Omega \nabla \mathbf{v}_1 : \nabla \mathbf{v}_2 \, d\mathbf{x}, \quad \forall (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega),$$

et la forme trilinéaire

$$b(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \int_\Omega (\mathbf{v}_1 \cdot \nabla) \mathbf{v}_2 \cdot \mathbf{v}_3 \, d\mathbf{x}, \quad \forall (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega).$$

En intégrant par parties la forme trilinéaire $b(\cdot,\cdot,\cdot)$, on obtient les propriétés suivantes :

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = \frac{\alpha^2}{2} \int_{\Gamma_b} |\mathbf{g}|^2 (\mathbf{u} \cdot \mathbf{n}) \, d\zeta, \ \forall \mathbf{u} \in \mathbf{V}(\Omega), \ \forall (\mathbf{v}, \alpha) \in W(Q), \tag{43}$$

$$b(\mathbf{v}, \mathbf{v}, \mathbf{v}) = \frac{\alpha^3}{2} \int_{\Gamma_b} |\mathbf{g}|^2 (\mathbf{g} \cdot \mathbf{n}) \, d\zeta, \ \forall (\mathbf{v}, \alpha) \in W(Q). \tag{44}$$

En plus, grâce à [20, Lemma 1.1, page 6] on a

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{u})| \leq C_\Omega \|\nabla \mathbf{v}\| \|\nabla \mathbf{u}\|^2, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \ \mathbf{u} \in \mathbf{H}^1_0(\Omega), \tag{45}$$

où

$$C_\Omega = \begin{cases} \frac{2\sqrt{2}|\Omega|^{1/6}}{3} & \text{si } d = 3 \\ \frac{|\Omega|^{1/2}}{2} & \text{si } d = 2. \end{cases} \tag{46}$$

Problème de stabilisation. Afin de stabiliser le système (41), nous choisissons de chercher la solution $\mathbf{v}$ sous la forme $\mathbf{v} = \mathbf{z} + \alpha \mathbf{w}$, où $\mathbf{z} \in \mathbf{V}_0(\Omega)$, $\mathbf{w}$ vérifie (28) et grâce aux techniques d'estimation a priori α satisfait :

$$\int_{\Gamma_b} [\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}] \cdot \mathbf{g} \, d\zeta = f(\mathbf{v}, \alpha), \tag{47}$$

où

$$f(\mathbf{v}, \alpha) = A_{\mathbf{z}} + B_s \alpha + a_b \alpha^2 + b_b \alpha - \lambda_\nu \left(2\langle \mathbf{w}, \mathbf{z} \rangle + \alpha \|\mathbf{w}\|^2 \right) - K\alpha \|\mathbf{v}\|^2, \tag{48}$$

avec

$$a_b = \frac{1}{2} \int_{\Gamma_b} |\mathbf{g}|^2 (\mathbf{g} \cdot \mathbf{n}), \qquad b_b = \frac{1}{2} \int_{\Gamma_b} |\mathbf{g}|^2 (\boldsymbol{\psi} \cdot \mathbf{n}),$$

$$A_{\mathbf{z}} = b(\mathbf{w}, \boldsymbol{\psi}, \mathbf{z}) + b(\mathbf{z}, \boldsymbol{\psi}, \mathbf{w}), \qquad B_s = b(\mathbf{w}, \boldsymbol{\psi}, \mathbf{w}).$$

Notons que dans (48), les constantes λ_ν et K sont strictement positives, tandis que les fonctions b_b, $A_{\mathbf{z}}$ et B_s dépendent du temps. Le choix de $\mathbf{v}$ sous la forme $\mathbf{v} = \mathbf{z}+\alpha\mathbf{w}$ entraîne $\mathbf{v} = \alpha\mathbf{g}$ sur Γ_b car $\mathbf{z} = 0$ sur Γ. En couplant le système (41) avec l'équation (47), le couple $(\mathbf{v}, p)$ satisfait maintenant le problème de stabilisation suivant :

$$\begin{cases} (a) & \dfrac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\boldsymbol{\psi} + (\boldsymbol{\psi}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0 & \text{dans } Q, \\ (b) & \nabla\cdot\ \mathbf{v} = 0 & \text{dans } Q, \\ (c) & \mathbf{v} = \alpha(t)\mathbf{g}(\mathbf{x}), & \text{sur } \Sigma_b, \\ (d) & \mathbf{v} = 0 & \text{sur } \Sigma_l, \\ (e) & \mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{dans } \Omega, \\ (f) & \displaystyle\int_{\Gamma_b} [\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}]\cdot\mathbf{g}\, d\zeta = f(\mathbf{v},\alpha). & \end{cases} \tag{49}$$

Notons que le contrôle α est a priori inconnu et satisfait une loi de feedback non linéaire grâce à l'équation (49-f). Dans le but de déterminer α, conduisant à la détermination du contrôle frontière $\mathbf{v}_b = \alpha\mathbf{g}$, le système (49) est résolu via une procédure de Galerkin qui consiste à construire une suite de solutions approchées en utilisant une base de Galerkin adéquate. Un résultat de compacité nous permet ensuite de passer à la limite dans le système nonlinéaire satisfait par les solutions approchées.

Formulation variationnelle. Nous considérons la formulation variationnelle du problème de stabilisation (49).

Définition 5.7. *Soit $T > 0$ un nombre réel arbitraire, nous dirons que $(\mathbf{v}, \alpha)$ est solution faible de (49) sur $[0, T)$ si*

- $\mathbf{v} \in [L^\infty(0, T; \mathbf{H}(\Omega)) \cap L^2(0, T; \mathbf{V}(\Omega))]$,
- $\exists\alpha \in L^\infty(0, T)$ *tel que* $\mathbf{v} = \alpha\mathbf{g}$ *sur* Γ_b,

$$\begin{cases} (a) & \langle d_t\mathbf{v}, \widetilde{\mathbf{v}}\rangle + \nu a(\mathbf{v}, \widetilde{\mathbf{v}}) + b(\mathbf{v}, \boldsymbol{\psi}, \widetilde{\mathbf{v}}) + b(\mathbf{v}_s, \mathbf{v}, \widetilde{\mathbf{v}}) + b(\mathbf{v}, \mathbf{v}, \widetilde{\mathbf{v}}) = \widetilde{\alpha} f(\mathbf{v}, \alpha), \\ (b) & \left(\displaystyle\int_\Omega \mathbf{v}\cdot\tilde{\mathbf{v}}\, d\mathbf{x}\right)(0) = \displaystyle\int_\Omega \mathbf{v}_0\cdot\tilde{\mathbf{v}}\, d\mathbf{x}, \end{cases} \tag{50}$$

$\forall(\widetilde{\mathbf{v}}, \widetilde{\alpha}) \in W(Q)$.

La condition initiale (50-b) a sens car pour toute solution de (50-a), on voit que la fonction $t \longrightarrow \int_\Omega \mathbf{v}(t)\cdot\tilde{\mathbf{v}}\, d\mathbf{x}$ est continue (voir [14] Corollaire II.4.2).

Résultat de stabilité.

Théorème 5.8. *Supposons que la vitesse initiale* $\mathbf{v}_0$ *et le profil* $\mathbf{g}$ *satisfont respectivement*

$$\mathbf{v}_0 \in \mathbf{H}(\Omega), \quad (\mathbf{v}_0 \cdot \mathbf{n})\,\mathbf{n} \in \mathbf{H}^{1/2}(\Gamma_b), \tag{51}$$

$$\mathbf{g} \in \mathbf{V}^{1/2}(\Gamma_b) \quad \textit{and} \quad \alpha_0\,\mathbf{g} \cdot \mathbf{n} = \mathbf{v}_0 \cdot \mathbf{n} \;\; \textit{on} \;\; \Gamma_b \;\textit{with}\; \mathbf{g} \cdot \mathbf{n} \neq 0,\, \alpha_0 \in \mathbb{R}. \tag{52}$$

Pour toute condition initiale $\mathbf{v}_0$*, arbitraire et satisfaisant (51), il existe une solution* $(\mathbf{v}, \alpha)$ *dans le sens de la définition 5.7, et une distribution* p *sur* Ω *tel que (49) soit vérifié. En plus,* $\mathbf{v}$ *satisfait les estimations suivantes :*

$$\|\mathbf{v}(t)\| \;\leq\; \|\mathbf{v}_0\|\, e^{-\sigma(t)}, \;\forall t > 0, \tag{53}$$

$$\int_0^T \|\nabla \mathbf{v}(t)\|^2 dt \;\leq\; C\|\mathbf{v}_0\|^2, \tag{54}$$

où $C > 0$ *est constant et pour tout* $K > 0$ *fixé, la fonction* $\sigma(t)$ *est définie comme suit :*

$$\sigma(t) = \lambda_1 \beta_\nu t + K \int_0^t \alpha^2(s) ds. \tag{55}$$

Remarque 5.9. *Dans (55) la constante positive* λ_1 *est la plus petite valeur propre de (25) et grâce à (39),* $\beta_\nu = \nu - C_\Omega \sup_{t \leq T} \|\nabla \boldsymbol{\psi}(t, \mathbf{x})\|$ *est un nombre réel strictement positif. En plus, le taux de décroissance* $\sigma(t) > 0$ *dépend du contrôle* α*.*

Remarque 5.10. *Puisque* $\beta_\nu > 0$*, la cible* $\boldsymbol{\psi}$ *est naturellement stable dans le sens où, si* α *est identiquement nul* ($\alpha \equiv 0$)*, le système (49) se stabilise seul. Cependant, en plus de (51)-(52), si la condition initiale* $\mathbf{v}_0$ *et le profil* $\mathbf{g}$ *sont tels que* $\alpha_0 \mathbf{g} \cdot \mathbf{n} = \mathbf{v}_0 \cdot \mathbf{n} \neq 0$ *sur* Γ_b*, par exemple, le contrôle* α *n'est pas identiquement nul.*

5.3 Stabilisation de type « feedback » du système de Navier-Stokes avec des conditions aux limites mixtes

On considère un domaine ouvert Ω de $\mathbb{R}^d$ ($d = 2$ ou $d = 3$), borné connexe de classe C^2 et de frontière Γ. Celle-ci est constituée de trois composantes connexes Γ_l, Γ_e et Γ_s tel que $\Gamma = \Gamma_l \cup \Gamma_e \cup \Gamma_s$. En particulier, le bord Γ_e est la partie de Γ, où le contrôle frontière sous forme de feedback est déterminé. On considère le couple vitesse-pression $(\mathbf{v}_s, q_s)$ solution du système de Navier-Stokes stationnaire

$$\begin{cases} -\nu \Delta \mathbf{v}_s + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s + \nabla q_s = \mathbf{f}_s, \quad \nabla \cdot \mathbf{v}_s = 0 & \text{dans } \Omega, \\ \mathbf{v}_s = 0 & \text{sur } \Gamma_l, \\ \mathbf{v}_s = \boldsymbol{\psi}_e & \text{sur } \Gamma_e, \\ \nu \nabla \mathbf{v}_s \cdot \mathbf{n} - q_s \mathbf{n} = \boldsymbol{\psi}_s & \text{sur } \Gamma_s, \end{cases} \tag{56}$$

où $\nu > 0$ est le coefficient de viscosité, $\mathbf{f}_s$ représente les forces massiques s'exerçant dans le fluide, $\boldsymbol{\psi}_e$ est la condition de Dirichlet sur Γ_e et $\boldsymbol{\psi}_s$ est la condition de Neumann sur Γ_s. En plus, nous supposons que $(\mathbf{v}_s, q_s)$ appartient à $\mathbf{H}^1(\Omega) \times L^2_0(\Omega)$.

Soient $Q = [0, T[\times\Omega$, $\Sigma_l = [0, T[\times\Gamma_l$, $\Sigma_e = [0, T[\times\Gamma_e$ et $\Sigma_s = [0, T[\times\Gamma_s$, on considère le couple $(\mathbf{u}, q)$, solution du problème de Navier-Stokes non-stationnaire suivant

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla q = \mathbf{f}_s, \quad \nabla\cdot\mathbf{u} = 0 & \text{dans } Q, \\ \mathbf{u}(\mathbf{x}) = 0 & \text{sur } \Sigma_l, \\ \mathbf{u}(t,\mathbf{x}) = \mathbf{u}_e(t,\mathbf{x}) + \boldsymbol{\psi}_e(\mathbf{x}) & \text{sur } \Sigma_e, \\ \nu\nabla\mathbf{u}\cdot\mathbf{n} - q\mathbf{n} = \mathbf{u}_s(t,\mathbf{x}) + \boldsymbol{\psi}_s(\mathbf{x}) & \text{sur } \Sigma_s, \\ \mathbf{u}(t=0,\mathbf{x}) = \mathbf{v}_s(\mathbf{x}) + \mathbf{v}_0(\mathbf{x}) & \text{dans } \Omega, \end{cases} \tag{57}$$

où $\mathbf{v}_0(\mathbf{x})$ est considéré comme une perturbation de l'état stationnaire $\mathbf{v}_s$. En substituant $\mathbf{u} = \mathbf{v} + \mathbf{v}_s$ et $q = p + q_s$ dans (57), le système du couple $(\mathbf{v}, p)$ qui en résulte s'écrit :

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}_s + (\mathbf{v}_s\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0 & \text{dans } Q, \\ \nabla\cdot\ \mathbf{v} = 0 & \text{dans } Q, \\ \mathbf{v} = 0 & \text{sur } \Sigma_l, \\ \mathbf{v}(t,\mathbf{x}) = \mathbf{u}_e(t,\mathbf{x}) & \text{sur } \Sigma_e, \\ \nu\nabla\mathbf{v}\cdot\mathbf{n} - p\mathbf{n} = \mathbf{u}_s(t,\mathbf{x}) & \text{sur } \Sigma_s, \\ \mathbf{v}(t=0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{dans } \Omega. \end{cases} \tag{58}$$

L'objectif du chapitre 3 est de trouver un $\mathbf{u}_s$ adéquat sur Σ_s et un contrôle $\mathbf{u}_e$ sur Σ_e qui stabilisent le système (58).

Nous allons maintenant résumer les différentes parties du chapitre 3 et énoncer le résultat principal. Commençons par définir quelques espaces fonctionnels.

Espaces fonctionnels. On considère les espaces des fonctions à divergence nulle suivants :

$$\mathcal{V}(\Omega) = \{\mathbf{u} \in \mathcal{D}(\Omega),\ \nabla\cdot\mathbf{u} = 0\}, \tag{59}$$

$$\mathbf{V}_0(\Omega) = \text{la fermeture } \mathcal{V}(\Omega) \text{ dans } \mathbf{H}^1_0(\Omega), \tag{60}$$

$$\mathbf{V}(\Omega) = \left\{\mathbf{u} \in \mathbf{H}^1(\Omega)\ : \nabla\cdot\mathbf{u} = 0,\ \ \mathbf{u} = 0 \text{ sur } \Gamma_l\right\}, \tag{61}$$

$$\mathbf{Z}(\Omega) = \left\{\mathbf{u} \in \mathbf{H}^1(\Omega)\ : \nabla\cdot\mathbf{u} = 0,\ \ \mathbf{u} = 0 \text{ sur } \Gamma_l \cup \Gamma_e\right\}, \tag{62}$$

$$\mathbf{H}(\Omega) = \left\{\mathbf{u} \in \mathbf{L}^2(\Omega)\ : \nabla\cdot\mathbf{u} = 0,\ \ \mathbf{u}\cdot\mathbf{n} = 0 \text{ sur } \Gamma_l\right\}. \tag{63}$$

Remarque 5.11. *Puisque* $\mathbf{V}(\Omega)$ *et* $\mathbf{Z}(\Omega)$ *sont chacun un sous espace fermé de* $\mathbf{H}^1(\Omega)$*, nous avons, par définition*

$$\|\cdot\|_{\mathbf{V}(\Omega)} = \|\cdot\|_{\mathbf{Z}(\Omega)} = \|\cdot\|_{\mathbf{H}^1(\Omega)}.$$

Remarque 5.12. *Puisque* $\mathbf{Z}(\Omega)$ *est un sous espace fermé de* $\mathbf{H}^1(\Omega)$*,* $\mathbf{Z}(\Omega)$ *est donc un espace de Hilbert séparable. A ce titre, il admet une base orthonormale dénombrable* $(\mathbf{z}_n)_{n\in\mathbb{N}}$ *qui sera utilisée dans la suite.*

Définition 5.13. *Soit* $\Gamma_i \subset \Gamma$*, on désigne par* $\mathbf{V}^{1/2}(\Gamma_i)$ *le sous-espace de* $\mathbf{H}^{1/2}(\Gamma)$ *formé des fonctions définies dans* Γ_i *et dont l'extension par zéro sur* $\Gamma \setminus \Gamma_i$ *appartient à* $\mathbf{H}^{1/2}(\Gamma)$*. En plus, on définit*

$$W(Q) = \{(\mathbf{v}, \alpha) \in \mathbf{V}(\Omega) \times \mathbb{R},\ \textit{tel que } \mathbf{v} = \alpha \mathbf{g} \textit{ sur } \Gamma_e\} \tag{64}$$

où g *satisfait*

$$\mathbf{g} \in \mathbf{V}^{1/2}(\Gamma_e), \quad \mathbf{g}\cdot\mathbf{n} \neq 0 \textit{ sur } \Gamma_e, \quad \int_{\Gamma_e} \mathbf{g}\cdot\mathbf{n}\, d\zeta = 0. \tag{65}$$

Remarque 5.14. *La solution de (58) est cherchée dans l'espace fonctionnel* $W(Q)$*, lequel est défini dans (64).*

Base de Galerkin pour $W(Q)$**.** Dans le cas où la fonction g satisfait les conditions (80), on considère ce problème de Stokes

$$\begin{cases} (a) & -\Delta \mathbf{w} + \nabla q = 0 & \text{dans } \ \Omega, \\ (b) & \nabla\cdot\mathbf{w} = 0 & \text{dans } \ \Omega, \\ (c) & \mathbf{w} = 0 & \text{sur } \ \Gamma_l \cup \Gamma_s, \\ (d) & \mathbf{w} = \mathbf{g} & \text{sur } \ \Gamma_e. \end{cases} \tag{66}$$

Dans le cas contraire, la fonction g est construite en adoptant la démarche suivante : nous supposons que le bord Γ_e est constitué de deux composantes connexes Γ_0 et Γ_1 tel que $\Gamma_e = \Gamma_0 \cup \Gamma_1$. Ensuite, pour tout g_0, g_1 tels que

$$\begin{aligned} &\mathbf{g}_0 \in V^{1/2}(\Gamma_0) \quad \text{et} \quad \int_{\Gamma_1} \mathbf{g}_0\cdot\mathbf{n}\, d\zeta \neq 0, \\ &\mathbf{g}_1 \in V^{1/2}(\Gamma_1) \quad \text{et} \quad \mathbf{g}_1\cdot\mathbf{n} \neq 0 \ \text{ sur } \ \Gamma_1, \end{aligned}$$

on construit g tel que.

$$\mathbf{g} = \begin{cases} \beta\mathbf{g}_0 & \text{sur } \ \Gamma_0, \\ \mathbf{g}_1 & \text{sur } \ \Gamma_1, \end{cases} \tag{67}$$

où $\beta = -\dfrac{\int_{\Gamma_1} \mathbf{g}_1 \cdot \mathbf{n}\, d\zeta}{\int_{\Gamma_0} \mathbf{g}_0 \cdot \mathbf{n}\, d\zeta}$. La fonction g définie dans (67) satisfait alors (80), et on considére le problème de Stokes suivant

$$\begin{cases} (a) & -\Delta \mathbf{w} + \nabla q = 0 & \text{dans } \Omega, \\ (b) & \nabla \cdot \mathbf{w} = 0 & \text{dans } \Omega, \\ (c) & \mathbf{w} = 0 & \text{sur } \Gamma_l \cup \Gamma_s, \\ (d) & \mathbf{w} = \beta \mathbf{g}_0 & \text{sur } \Gamma_0, \\ (e) & \mathbf{w} = \mathbf{g}_1 & \text{sur } \Gamma_1. \end{cases} \tag{68}$$

Puisque $\mathbf{w} = \mathbf{g}$ sur $\Gamma_e = \Gamma_0 \cup \Gamma_1$, le système (66) ou (68) admet une solution unique $(\mathbf{w}, q)$ dans $\mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ (voir [14, Proposition III.4.1]). Par ailleurs, pour toute fonction $\mathbf{z} \in \mathbf{Z}(\Omega)$ définie dans (62) et pour tout $\alpha \in \mathbb{R}$, nous avons $\mathbf{v} = \mathbf{z} + \alpha \mathbf{w} \in W(\Omega)$. En effet, nous avons $\mathbf{z}, \mathbf{w} \in \mathbf{V}(\Omega)$ et puisque $\mathbf{z} = 0$ on Γ_e alors $\mathbf{v} = \alpha \mathbf{g}$ sur Γ_e. D'après la remarque 5.12, l'espace $\mathbf{Z}(\Omega)$ admet une base orthonormale $(\mathbf{z}_n)_{n \in \mathbb{N}}$. La suite $\mathbf{w},\ \mathbf{z}_1,\ \mathbf{z}_2,\ \mathbf{z}_3, \cdots$, est alors linéairement indépendante. Par conséquent, l'espace de la solution v du système (58) est réécrit comme suit :

$$W(Q) = \text{span}(\mathbf{w}) \oplus \text{span}(\mathbf{z}_n)_{\{n \in \mathbb{N}^*\}}. \tag{69}$$

Formes linéaires. Afin de définir la formulation faible du problème de stabilisation des équations de Navier-Stokes, on introduit la forme bilinéaire

$$a(\mathbf{v}_1, \mathbf{v}_2) = \int_\Omega \nabla \mathbf{v}_1 : \nabla \mathbf{v}_2 \, d\mathbf{x}, \quad \forall \mathbf{v}_j \in \mathbf{H}^1(\Omega),\ j = 1, 2,$$

et la forme trilinéaire

$$b(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \int_\Omega (\mathbf{v}_1 \cdot \nabla) \mathbf{v}_2 \cdot \mathbf{v}_3 \, d\mathbf{x}, \quad \forall \mathbf{v}_j \in \mathbf{H}^1(\Omega),\ j = 1, 2.$$

En intégrant par parties la forme trilinéaire $b(\cdot, \cdot, \cdot)$, on obtient les identités suivantes suivantes :

$$b(\mathbf{v}_s, \mathbf{v}, \mathbf{v}) = \frac{1}{2} \int_{\Gamma_s} |\mathbf{z}|^2 (\mathbf{v}_s \cdot \mathbf{n}) \, d\zeta + \frac{\alpha^2}{2} \int_{\Gamma_e} |\mathbf{g}|^2 (\mathbf{v}_s \cdot \mathbf{n}) \, d\zeta,\ \forall (\mathbf{v}, \alpha) \in W(Q), \tag{70}$$

$$b(\mathbf{v}, \mathbf{v}, \mathbf{v}) = \frac{1}{2} \int_{\Gamma_s} |\mathbf{z}|^2 (\mathbf{z} \cdot \mathbf{n}) \, d\zeta + \frac{\alpha^3}{2} \int_{\Gamma_e} |\mathbf{g}|^2 (\mathbf{g} \cdot \mathbf{n}) \, d\zeta,\ \forall (\mathbf{v}, \alpha) \in W(Q). \tag{71}$$

Grâce à l'inégalité de Hölder, la fonction b satisfait

$$|b(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)| \leq \|\mathbf{v}_1\|_{\mathbf{L}^3(\Omega)} \|\nabla \mathbf{v}_2\| \|\mathbf{v}_3\|_{\mathbf{L}^6(\Omega)}, \quad \forall \mathbf{v}_j \in \mathbf{H}^1(\Omega),\ j = 1, 2, 3.$$

En plus, d'après les inégalités de Sobolev généralisées, on a

$$\|\mathbf{v}_1\|_{\mathbf{L}^3(\Omega)} \leq C \|\mathbf{v}_1\|^{\frac{1}{2}} \|\nabla \mathbf{v}_1\|^{\frac{1}{2}} \quad \text{and} \quad \|\mathbf{v}_3\|_{\mathbf{L}^6(\Omega)} \leq C \|\nabla \mathbf{v}_3\|, \quad \text{for } d = 2, 3,$$

où C est une constante positive. On obtient, alors

$$|b(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)| \leq C \|\mathbf{v}_1\|^{\frac{1}{2}} \|\nabla \mathbf{v}_1\|^{\frac{1}{2}} \|\nabla \mathbf{v}_2\| \|\nabla \mathbf{v}_3\|. \tag{72}$$

Problème de stabilisation. Avant de donner le problème de stabilisation, nous allons définir la loi de contrôle et la condition de Neumann $\mathbf{u}_s$ sur Σ_s. Rappelons que la solution $\mathbf{v}$ est cherchée sous la forme $\mathbf{v} = \mathbf{z} + \alpha \mathbf{w}$, où $\mathbf{z} \in \mathbf{Z}(\Omega)$, $\alpha \in \mathbb{R}$ et $\mathbf{w}$ vérifie (66) ou (68). Grâce aux techniques d'estimation a priori, le contrôle α satisfait :

$$\int_{\Gamma_b} [\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}] \cdot \mathbf{g}\, d\zeta = \mathcal{F}(\mathbf{v}, \alpha), \tag{73}$$

où

$$\mathcal{F}(\mathbf{v}, \alpha) = a_e \alpha^2 + b_e \alpha - \lambda_\nu \left(\alpha \|\mathbf{w}\|^2 + 2\langle \mathbf{w}, \mathbf{z} \rangle \right) + 2\, \beta_\nu \langle \nabla \mathbf{w}, \nabla \mathbf{z} \rangle - K\alpha \|\mathbf{v}\|^2, \tag{74}$$

avec $\lambda_\nu, \beta_\nu > 0$,

$$a_e = \frac{1}{2} \int_{\Gamma_e} |\mathbf{g}|^2 (\mathbf{g} \cdot \mathbf{n})\, d\zeta \quad \text{et} \quad b_e = \frac{1}{2} \int_{\Gamma_e} |\mathbf{g}|^2 (\mathbf{v}_s \cdot \mathbf{n})\, d\zeta.$$

Concernant la condition de Neumann $\mathbf{u}_s$ sur Σ_s, rappelons que si l'écoulement est sortant sur Γ_s, $\mathbf{v} \cdot \mathbf{n} > 0$ et s'il est entrant, $\mathbf{v} \cdot \mathbf{n} < 0$. Par ailleurs, puisque par construction, $\mathbf{w} = 0$ sur Γ_s, on a $\mathbf{v} = \mathbf{z}$ sur Γ_s. En plus, la fonction $\mathbf{z}$ est inconnue sur Γ_s. Pour tenir compte des deux cas : écoulement rentrant \ écoulement sortant, nous prenons la condition aux limites absorbante (voir [14, Page 247]) sous la forme

$$\mathbf{u}_s = -\frac{1}{2} \mathbf{z} \left[(\mathbf{v}_s \cdot \mathbf{n})^- + (\mathbf{z} \cdot \mathbf{n})^- \right] \quad \text{sur} \quad \Gamma_s. \tag{75}$$

En rappelant que les parties positives et négatives de tout réel x, sont définies par $x^+ = \max(x, 0)$, $x^- = \min(x, 0)$, de sorte que l'on a $x = x^+ - x^-$, la condition (75) est déduite de (70)-(71), en utilisant seulement les termes en $\mathbf{z}$. En couplant le système (58) avec l'équation (73) et la condition (75), le couple $(\mathbf{v}, p)$ satisfait maintenant le problème de stabilisation suivant :

$$\begin{cases} (a) & \dfrac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}_s + (\mathbf{v}_s\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0 & \text{dans } Q, \\ (b) & \nabla\cdot\ \mathbf{v} = 0 & \text{dans } Q, \\ (c) & \mathbf{v} = 0 & \text{sur } \Sigma_l, \\ (d) & \mathbf{v} = \alpha(t)\mathbf{g}(\mathbf{x}) & \text{sur } \Sigma_e, \\ (e) & \nu\nabla\mathbf{v}\cdot\mathbf{n} - p\mathbf{n} = -\dfrac{1}{2}\,\mathbf{z}\left[(\mathbf{v}_s\cdot\mathbf{n})^- + (\mathbf{z}\cdot\mathbf{n})^-\right] & \text{sur } \Sigma_s, \\ (f) & \displaystyle\int_{\Gamma_e}[\nu\nabla\mathbf{v}\cdot\mathbf{n} - p\mathbf{n}]\cdot\mathbf{g}\,d\zeta = \mathcal{F}(\mathbf{v},\alpha), & \\ (g) & \mathbf{v}(t=0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{dans } \Omega. \end{cases} \tag{76}$$

Notons qu'ici encore, le contrôle α est a priori inconnu et satisfait une loi de feedback non linéaire grâce à l'équation (76-f). Dans le but de déterminer α, conduisant à la détermination du contrôle frontière $\mathbf{v}_b = \alpha\mathbf{g}$, le système (76) est résolu via une procédure de Galerkin qui consiste à construire une suite de solutions approchées en utilisant une base de Galerkin adéquate. Un résultat de compacité nous permet ensuite de passer à la limite dans le système non-linéaire satisfait par les solutions approchées.

Formulation variationnelle. En intégrant par parties sur Ω le problème de stabilisation (76), nous obtenons une formulation faible qui conduit à la définition suivante

Définition 5.15. *Soit $T > 0$ un nombre réel arbitraire et $\mathbf{v}_0 \in \mathbf{H}(\Omega)$, nous dirons que $(\mathbf{v},\alpha)$ est solution faible de (76) sur $[0,T)$ si*

(i) $\mathbf{v} \in [L^\infty(0,T;\mathbf{H}(\Omega)) \cap L^2(0,T;\mathbf{V}(\Omega))]$,

(ii) $\alpha \in L^\infty(0,T)$ *tel que* $\mathbf{v}(t,\mathbf{x}) = \alpha(t)\mathbf{g}(\mathbf{x})$ *sur* Γ_e,

(iii) $\forall\,\widetilde{\mathbf{v}} = \widetilde{\mathbf{z}} + \widetilde{\alpha}\mathbf{w} \in W(Q)$, *la formulation variationnelle suivante est satisfaite*

$$\begin{cases} (a) & \langle d_t\mathbf{v},\widetilde{\mathbf{v}}\rangle + \nu a(\mathbf{v},\widetilde{\mathbf{v}}) + b(\mathbf{v},\mathbf{v}_s,\widetilde{\mathbf{v}}) + b(\mathbf{v}_s,\mathbf{v},\widetilde{\mathbf{v}}) + b(\mathbf{v},\mathbf{v},\widetilde{\mathbf{v}}) \\ & \qquad = \widetilde{\alpha}\mathcal{F}(\mathbf{v},\alpha) - \dfrac{1}{2}\displaystyle\int_{\Gamma_s}(\mathbf{z}\cdot\widetilde{\mathbf{z}})\big((\mathbf{v}_s\cdot\mathbf{n})^- + (\mathbf{z}\cdot\mathbf{n})^-\big), \\ (b) & \left(\displaystyle\int_\Omega \mathbf{v}\cdot\widetilde{\mathbf{v}}\,d\mathbf{x}\right)(0) = \displaystyle\int_\Omega \mathbf{v}_0\cdot\widetilde{\mathbf{v}}\,d\mathbf{x}. \end{cases} \tag{77}$$

Ici aussi, la condition initiale (77-b) a du sens car pour toute solution de (50-a), on voit que la fonction $t \longrightarrow \int_\Omega \mathbf{v}(t)\cdot\tilde{\mathbf{v}}\,d\mathbf{x}$ est continue.

La principale réalisation du chapitre 3, est le résultat de stabilisation suivant.

Résultat de stabilité.

Théorème 5.16. *Supposons que l'état stationnaire* $\mathbf{v}_s$ *solution de (56) satisfait*

$$\beta_\nu = \nu - C_2 \|\nabla \mathbf{v}_s\| > 0, \tag{78}$$

où la constante $C_2 > 0$ *est définie dans (3.36). Supposons que la vitesse initiale* $\mathbf{v}_0$ *et le profil* $\mathbf{g}$ *satisfont respectivement*

$$\mathbf{v}_0 \in \mathbf{H}(\Omega), \quad (\mathbf{v}_0 \cdot \mathbf{n})\,\mathbf{n} \in \mathbf{H}^{1/2}(\Gamma_e), \tag{79}$$

$$\mathbf{g} \in \mathbf{V}^{1/2}(\Gamma_e) \quad \textit{and} \quad \alpha_0\,\mathbf{g} \cdot \mathbf{n} = \mathbf{v}_0 \cdot \mathbf{n} \;\; \textit{on} \;\; \Gamma_e \;\; \textit{with} \;\; \mathbf{g} \cdot \mathbf{n} \neq 0,\, \alpha_0 \in \mathbb{R}. \tag{80}$$

Pour toute condition initiale $\mathbf{v}_0$ *arbitraire et satisfaisant (79), il existe une solution faible* $(\mathbf{v}, \alpha)$ *dans le sens de la définition 5.15, et une distribution* p *sur* Ω *tel que (76) soit vérifié. En plus, il existe une constante positive* σ *tel que* $\mathbf{v}$ *satisfait*

$$\|\mathbf{v}\| \leq \|\mathbf{v}_0\| \exp\left(-\sigma t - K \int_0^t \alpha^2(s) ds\right), \tag{81}$$

où la constante $K > 0$ *est fixée. En outre,*

$$\int_0^T \|\nabla \mathbf{v}\|^2 \leq C_\nu \|\mathbf{v}_0\|^2, \tag{82}$$

où la constant C_ν *dépend de* ν*.*

Remarque 5.17. *Avec la condition (78), la cible* $\mathbf{v}_s$ *est naturellement stable dans le sens où, si* $\alpha \equiv 0$ *et* $\mathbf{z} \equiv 0$ *sur* Γ_s*, le système (76) se stabilise seul. Dans le cas où* $\mathbf{z} \equiv 0$ *sur* Γ_s*, en plus de (79)-(80), si la condition initiale* $\mathbf{v}_0$ *et le profil* $\mathbf{g}$ *sont tels que* $\alpha_0 \mathbf{g} \cdot \mathbf{n} = \mathbf{v}_0 \cdot \mathbf{n} \neq 0$ *sur* Γ_b*, par exemple, le contrôle* α *n'est pas identiquement nul.*

5.4 Méthode des caractéristiques-Galerkin pour le contrôle frontière des équations de Navier-Stokes

On considère un domaine ouvert Ω de $\mathbb{R}^d$ ($d = 2$ ou $d = 3$), borné connexe de classe C^2 et de frontière $\partial\Omega = \Gamma$. Celle-ci est constituée de trois composantes connexes Γ_l, Γ_b et Γ_s tel que $\Gamma = \Gamma_l \cup \Gamma_b \cup \Gamma_s$. En particulier, le bord Γ_b est la partie de Γ où le contrôle frontière sous forme de feedback est déterminé. On considère dans Ω, un écoulement incompressible stationnaire décrit par le couple $(\mathbf{v}_s, q_s)$, solution système de Navier-Stokes suivant

$$\begin{cases} -\nu\Delta \mathbf{v}_s + (\mathbf{v}_s.\nabla)\mathbf{v}_s + \nabla q_s = \mathbf{f}_s & \text{dans } \Omega, \\ \nabla \cdot \mathbf{v}_s = 0 & \text{dans } \Omega, \\ \mathbf{v}_s = \mathbf{v}_b & \text{sur } \Gamma_b, \\ \mathbf{v}_s = 0 & \text{sur } \Gamma_l, \\ \nu \nabla \mathbf{v}_s \cdot \mathbf{n} - q_s \mathbf{n} = 0 & \text{sur } \Gamma_s, \end{cases} \tag{83}$$

où $\nu > 0$ est la viscosité, $\mathbf{f}_s$ le champ de force et $\mathbf{v}_b$ la condition au bord sur Γ_b.

Pour tout $T > 0$ fixé, on pose $Q = [0,T[\times\Omega$, $\Sigma_l = [0,T[\times\Gamma_l$, $\Sigma_b = [0,T[\times\Gamma_b$ et $\Sigma_s = [0,T[\times\Gamma_s$ et on considère le problème de Navier-Stokes non-stationnaire

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu\Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla q = \mathbf{f}_s & \text{dans } Q, \\ \nabla \cdot \mathbf{u} = 0 & \text{dans } Q, \\ \mathbf{u} = \mathbf{v}_b + \mathbf{u}_b & \text{sur } \Sigma_b, \\ \mathbf{u} = 0 & \text{sur } \Sigma_l, \\ \nu \nabla \mathbf{u} \cdot \mathbf{n} - q \mathbf{n} = 0 & \text{sur } \Sigma_s, \\ \mathbf{u}_0(\mathbf{x}) = \mathbf{v}_s(\mathbf{x}) + \mathbf{v}_0(\mathbf{x}) & \text{dans } \Omega, \end{cases} \tag{84}$$

où $\mathbf{u}_b$ représente le contrôle et $\mathbf{v}_0$ la perturbation de l'état initiale.

En remplaçant $(\mathbf{u}, q) = (\mathbf{v} + \mathbf{v}_s, p + q_s)$ dans (84), on obtient le système

$$\begin{cases} (a) \quad \dfrac{\partial \mathbf{v}}{\partial t} - \nu\Delta \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}_s + (\mathbf{v}_s\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0 & \text{dans } Q, \\ (b) \quad \nabla \cdot \mathbf{v} = 0 & \text{dans } Q, \\ (c) \quad \mathbf{v} = \mathbf{u}_b & \text{sur } \Sigma_b, \\ (d) \quad \mathbf{v} = 0 & \text{sur } \Sigma_l, \\ (e) \quad \nu \nabla \mathbf{v} \cdot \mathbf{n} - p \mathbf{n} = 0 & \text{sur } \Sigma_s, \\ (f) \quad \mathbf{v}(t=0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{dans } \Omega. \end{cases} \tag{85}$$

Nous allons utiliser la méthode des caractéristiques pour définir le problème de stabilisation discret en temps, correspondant au système (85).

Discrétisation en temps du problème de stabilisation. Soient $X(\tau;t,\mathbf{x})$ et $Y(\tau;t,\mathbf{x})$, les solutions en τ des équations différentielles ordinaires

$$\begin{cases} (a) \quad \dfrac{dX}{d\tau} = \mathbf{v}(\tau, X(\tau;t,\mathbf{x})) & \text{si } X(\tau;t,\mathbf{x}) \in \Omega, \\ \qquad\quad = 0 & \text{sinon}, \\ (b) \quad X(t;t,\mathbf{x}) = \mathbf{x}, & \end{cases} \tag{86}$$

$$\begin{cases} (a) \quad \dfrac{dY}{d\tau} = \mathbf{v}(\tau, Y(\tau;t,\mathbf{x})) + 2\,\mathbf{v}_s(Y(\tau;t,\mathbf{x})) & \text{si } Y(\tau;t,\mathbf{x}) \in \Omega, \\ \qquad\quad = 0 & \text{sinon}, \\ (b) \quad Y(t;t,\mathbf{x}) = \mathbf{x}. & \end{cases} \tag{87}$$

Dans ces équations, $X(\cdot\,;\,t,\mathbf{x})$ ou $Y(\cdot\,;\,t,\mathbf{x})$ représente la position de la particule à l'instant τ qui se trouve au point $\mathbf{x} = (x_1, x_2, x_3)$ au temps t. Lorsque $t^0 = 0 < t^1 < t^2 < \cdots < t^N = T$, les pieds des caractéristiques $X(t^{n-1};t^n,\mathbf{x})$ et $Y(t^{n-1};t^n,\mathbf{x})$ sont calculés à partir de (86) et de (87), respectivement :

$$X(t^{n-1};t^n,\mathbf{x}) \approx \mathbf{x} - \mathbf{v}(t^n,\mathbf{x})\Delta t,$$
$$Y(t^{n-1};t^n,\mathbf{x}) \approx \mathbf{x} - \mathbf{u}(t^n,\mathbf{x})\Delta t,$$

où $\mathbf{u} = \mathbf{v} + 2\,\mathbf{v}_s$ et le pas de temps $\Delta t = t^n - t^{n-1} = T/N$. En plus, grâce à (86-b) et (87-b), nous avons $X(t^n;t^n,\mathbf{x}) = Y(t^n;t^n,\mathbf{x}) = \mathbf{x}$.

En posant

$$\mathbf{v}^n = \mathbf{v}(t^n,\mathbf{x}), \qquad p^n = p(t^n,\mathbf{x}), \qquad X^n = \mathbf{x} - \mathbf{v}(t^n,\mathbf{x})\Delta t \qquad \text{et} \qquad Y^n = \mathbf{x} - \mathbf{u}(t^n,\mathbf{x})\Delta t,$$

la discrétisation en temps de (85) par la méthode des caractéristiques, conduit à

$$\begin{cases} (a) \quad \dfrac{\mathbf{v}^n}{\Delta t} - \nu\Delta\mathbf{v}^n + \nabla p^n = \dfrac{F^{n-1}}{\Delta t} & \text{dans } \Omega, \\ (b) \quad \nabla\cdot\mathbf{v}^n = 0 & \text{dans } \Omega, \\ (c) \quad \mathbf{v}^n = 0 & \text{sur } \Gamma_l, \\ (d) \quad \mathbf{v}^n = \alpha_n \mathbf{g}(\mathbf{x}) & \text{sur } \Gamma_b, \\ (e) \quad \nu\nabla\mathbf{v}^n\cdot\mathbf{n} - p^n\mathbf{n} = 0 & \text{sur } \Sigma_s, \\ (f) \quad \mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{dans } \Omega, \end{cases} \tag{88}$$

où

$$F^{n-1} = \frac{1}{2}\big(\mathbf{v}^{n-1} \circ X^{n-1} + \mathbf{v}^{n-1} \circ Y^{n-1}\big) + \mathbf{v}_s \circ X^{n-1} - \mathbf{v}_s, \tag{89}$$

avec $\mathbf{v} \circ Z$ représentant la fonction $\mathbf{x} \to \mathbf{v}[Z(\mathbf{x})]$.

Le but du chapitre 4 est de trouver un contrôle α_n tel que $\mathbf{u}_b = \alpha_n \mathbf{g}(\mathbf{x})$ sur Σ_b stabilise le problème (88). Avant d'énoncer les résultats de stabilisation obtenus, nous commençons par résumer les parties essentielles de ce chapitre.

Processus de construction de la loi de contrôle. Puisque le système (88) est linéaire, la solution $(\mathbf{v}^n, p^n)$ est décomposée comme suit

$$\begin{cases} \mathbf{v}^n &= \widetilde{\mathbf{w}}^n + \alpha_n \mathbf{w}, \\ p^n &= \widetilde{q}^n + \alpha_n q, \end{cases} \tag{90}$$

où $(\mathbf{w}, q)$ ne dépend pas du temps, alors que $(\widetilde{\mathbf{w}}^n, \widetilde{q}^n)$ représente le terme de correction calculé à chaque instant. Les détails du processus de construction du contrôle sont décrits de la manière suivante :

(i) Premièrement, nous cherchons $(\mathbf{w}, q)$ tel que

$$\begin{cases} (a) & \dfrac{\mathbf{w}}{\Delta t} - \nu\Delta\mathbf{w} + \nabla q = 0 & \text{in } \ \Omega, \\ (b) & \nabla \cdot \mathbf{w} = 0 & \text{in } \ \Omega, \\ (c) & \mathbf{w} = 0 & \text{on } \ \Gamma_l, \\ (d) & \mathbf{w} = \mathbf{g} & \text{on } \ \Gamma_b \\ (e) & \nu\nabla\mathbf{w} \cdot \mathbf{n} - q\mathbf{n} = 0 & \text{on } \ \Gamma_s. \end{cases} \tag{91}$$

(ii) Deuxièmement, à chaque instant, nous cherchons $(\widetilde{\mathbf{w}}^n, \widetilde{q}^n)$ solution de

$$\begin{cases} (a) & \dfrac{\widetilde{\mathbf{w}}^n}{\Delta t} - \nu\Delta\widetilde{\mathbf{w}}^n + \nabla\widetilde{q}^n = F^{n-1} & \text{in } \ \Omega, \\ (b) & \nabla \cdot \widetilde{\mathbf{w}}^n = 0 & \text{in } \ \Omega, \\ (c) & \widetilde{\mathbf{w}}^n = 0 & \text{on } \ \Gamma_l \cup \Gamma_b \\ (d) & \nu\nabla\widetilde{\mathbf{w}}^n \cdot \mathbf{n} - \widetilde{q}^n\mathbf{n} = 0 & \text{on } \ \Gamma_s. \end{cases} \tag{92}$$

(iii) Enfin, dans le but de stabiliser (88) avec $\mathbf{v}^n = \alpha_n \mathbf{g}(\mathbf{x})$ sur Γ_b, en utilisant les techniques d'estimation a priori de l'énergie, la quantité α_n doit satisfaire, par exemple, la relation suivante

$$\int_{\Gamma_b} [\nu \nabla \mathbf{v}^n \cdot \mathbf{n} - p^n \mathbf{n}] \cdot \mathbf{g} = -\lambda \alpha_n, \quad \lambda > 0. \tag{93}$$

Pour tout $n \in \mathbb{N}$, nous supposons

$$X^n(\mathbf{x}) \;=\; \mathbf{x} - \mathbf{v}^n(\mathbf{x})\,\Delta t \in \Omega, \tag{94}$$

$$Y^n(\mathbf{x}) \;=\; \mathbf{x} - \mathbf{u}^n(\mathbf{x})\,\Delta t \in \Omega. \tag{95}$$

La formule de Taylor nous permet d'obtenir

$$\mathbf{v}_s\big(X^n(\mathbf{x})\big) = \mathbf{v}_s(\mathbf{x}) - \Delta t \nabla \mathbf{v}_s(\mathbf{x}) \cdot \mathbf{v}^n(\mathbf{x}) + O(\Delta t^2).$$

Ainsi, nous supposons

$$\mathbf{v}_s\big(X^n(\mathbf{x})\big) \;=\; \mathbf{v}_s(\mathbf{x}) - \Delta t \nabla \mathbf{v}_s(\mathbf{x}) \cdot \mathbf{v}^n(\mathbf{x}), \tag{96}$$

et énonçons les deux propositions suivantes

Proposition 5.1. *Soient $\mathbf{v}_0 \in \mathbf{H}(\Omega)$, $\mathbf{g} \in V^{\frac{1}{2}}(\Gamma_b)$ avec $\mathbf{g} \neq 0$ sur Γ_b et $\mathbf{v}_s$ tel que*

$$\|\nabla \mathbf{v}_s\| \leq \frac{1}{\Delta t}\left(\sqrt{1 + \frac{2\,\nu \Delta t}{C_p^2}} - 1\right), \tag{97}$$

où C_p est la constante de Poincaré. Sous les hypothèses (94)-(95) et (96), il existe un contrôle frontière α_n sur Γ_b solution de

$$\int_{\Gamma_b} [\nu \nabla \mathbf{v}^n \cdot \mathbf{n} - p^n \mathbf{n}] \cdot \mathbf{g} = -\lambda \alpha_n, \quad \lambda > 0 \tag{98}$$

tel que le système (88) avec $(\mathbf{v}^n, p^n)$ soit exponentiellement stable. i.e. il existe $\mu > 0$ tel que $\mathbf{v}^n$ satisfait

$$\|\mathbf{v}^n\| \;\leq\; \|\mathbf{v}_0\| \exp\left(-\mu t^n\right). \tag{99}$$

Remarque 5.18. *Dans la Proposition 5.1, la loi de contrôle (98) permet de trouver un contrôle α_n, solution d'un polynôme de degré un. Cependant, obtenir un contrôle optimal, en utilisant cette loi de contrôle, n'est pas évident. La proposition suivante permet ainsi de définir l'intervalle maximale dans lequel le contrôle appartient.*

Proposition 5.2. *Sous les hypothèses (94)-(96) et (97), il existe un θ dans $]0,1[$ tel que la solution $\widetilde{\mathrm{w}}^n$ de (92) satisfait*

$$\|\widetilde{\mathbf{w}}^n\| \leq \theta \|\mathbf{v}^{n-1}\|. \tag{100}$$

Par conséquence, il existe un contrôle frontière α_n, solution d'un polynôme de degré deux, rendant exponentiellement stable le système (88). i.e. il existe $\mu > 0$ tel que

$$\|\mathbf{v}^n\| \quad \leq \quad \|\mathbf{v}_0\| \exp\left(-\mu t^n\right). \tag{101}$$

Pour terminer ce chapitre, nous présenterons des résultats numériques dans le cas d'un écoulement autour d'un obstacle circulaire.

Nous allons donner la liste des travaux rassemblés dans cette thèse et présenter quelques perspectives ouvertes dans le contexte de la stabilisation frontière de certains systèmes hydrauliques.

6 Travaux en cours et perspectives

Liste des travaux rassemblés dans la thèse. Les différents travaux rassemblés dans cette thèse, en collaboration avec Abdou Sène et Daniel le Roux, ont fait l'objet des publications suivantes

- ⋆ Chapitre 1 : Boundary stabilization of the Navier-Stokes equations with feedback controller via a Galerkin method, paru dans *Evolution Equations and Control Theory, Volume 3, Pages 147-166, **2014***.
- ⋆ Chapitre 2 : Boundary stabilization of the Navier-Stokes Model with feedback controller around a non-stationary state, soumis.
- ⋆ Chapitre 3 : Feedback stabilization of the Navier-Stokes system with mixed boundary conditions, soumis.
- ⋆ Chapitre 4 : Numerical feedback stabilization of the Navier-Stokes equations using characteristic-Galerkin method. Le travail en cours sera soumis en juillet 2014.

Nous allons maintenant présenter quelques perspectives ouvertes par les travaux effectués dans cette thèse.

Contrôle en dimension fini N. Dans le théorème 5.3, un résultat de stabilité est obtenu pour une condition initiale arbitrairement choisie dans $\mathbf{H}(\Omega)$. Cependant, le taux de décroissance σ est fixé par la condition (33). Lorsque λ_N désigne la N-ième valeur propre de l'opérateur de Stokes défini dans (25), notre prochain objectif est d'essayer d'obtenir un taux de décroissance limité par λ_N i.e. $0 < \sigma \leq \nu\lambda_N - \|\nabla \mathbf{v}_s\|_\infty$. Ce résultat permettra non seulement d'augmenter le taux de décroissance, mais aussi de stabiliser une classe plus large d'états stationnaires $\mathbf{v}_s$. L'utilisation de l'opérateur de Oseen pourrait être envisagée car, dans la plupart des travaux cités, cet opérateur a permis d'obtenir un résultat semblable pour des conditions initiales assez petites.

Problème de Saint-Venant ou "Shallow water". Sans les forces de frottement et la force de Coriolis, le problème de Saint-Venant 2D, dans sa forme conservative, est caractérisé par le système suivant :

$$\begin{cases} (a) \quad \dfrac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + g\nabla h = 0 & \text{dans } Q, \\ (b) \quad \dfrac{\partial h}{\partial t} + \nabla \cdot (h\, \mathbf{u}) = 0 & \text{dans } Q, \\ + \text{ condition initiale et conditions aux bords} \end{cases} \tag{102}$$

où $\mathbf{u}$ représente la vitesse, h la hauteur du fluide, ν le coefficient de diffusion et g le coefficient de gravité. Ce problème décrit un écoulement à surface libre en eaux peu profondes. La stabilisation frontière du problème (102) n'a pas été abordée dans la littérature. Cependant, le cas linéaire à été traité dans [16] où les auteurs obtiennent un résultat de stabilité du système de Saint-Venant grâce à une méthode basée sur la symétrisation des matrices de flux du modèle linéarisé et l'analyse des invariants de Riemann. Pour stabiliser le modèle non-linéaire, nous aimerions utiliser la méthode proposée dans cette thèse.

Méthodes Numériques. Le problème de stabilisation tel que défini dans les trois premiers chapitres nécessite que l'on utilise à la fois deux conditions sur le même bord. Par exemple dans le système (76) nous avons :

$$\begin{cases} (a) \quad \mathbf{v} = \alpha(t)\mathbf{g}(\mathbf{x}) \;\; \textbf{sur} \;\; \Sigma_e = [0, T[\times \Gamma_e, \\ (b) \displaystyle\int_{\Gamma_e} [\nu \nabla \mathbf{v} \cdot \mathbf{n} - p\mathbf{n}] \cdot \mathbf{g}\, d\zeta = \mathcal{F}(\mathbf{v}, \alpha). \end{cases} \tag{103}$$

C'est la raison pour laquelle nous avons proposé dans cette thèse une approche numérique basée sur une méthode de Lagrange-Galerkin (ou méthode des caractéristiques). Celle-ci stabilise le problème de Navier-Stokes et peut être plus facilement implémentée. Cependant, la loi de contrôle numérique utilisée est différente de celle définie dans la théorie. Pour mieux consolider les résultats théoriques, nous envisageons d'utiliser d'autres approches numériques comme la méthode de Galerkin discontinue en espace et les méthodes explicites de Runge-Kutta en temps, appliquées aux équations de Navier-Stokes incompressibles.

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Chapitre 1

Boundary stabilization of the Navier-Stokes equations with feedback controller via a Galerkin method

Abstract

In this work we study the exponential stabilization of the two and three-dimensional Navier-Stokes equations in a bounded domain Ω, around a given steady-state flow, by means of a boundary control. In order to determine a feedback law, we consider an extended system coupling the Navier-Stokes equations with an equation satisfied by the control on the domain boundary. While most traditional approaches apply a feedback controller via an algebraic Riccati equation, the Stokes-Oseen operator or extension operators, a Galerkin method is proposed instead in this study. The Galerkin method permits to construct a stabilizing boundary control and by using energy a priori estimation technics, the exponential decay is obtained. A compactness result then allows us to pass to the limit in the system satisfied by the approximated solutions. The resulting feedback control is proven to be globally exponentially stabilizing the steady states of the two and three-dimensional Navier-Stokes equations.

Keywords : Navier-Stokes system, feedback control, boundary stabilization, Galerkin method.

1 Introduction

Let Ω be a bounded and connected domain in $\mathbb{R}^d$ $(d = 2, 3)$, with a boundary Γ of class C^2, and composed of two connected components Γ_l and Γ_b such that $\Gamma = \Gamma_l \cup \Gamma_b$, in order to impose two different boundary conditions specified in (1.1). In particular, the boundary Γ_b is the part of Γ, where a boundary control in feedback form has to be determined.

The usual function spaces $L^2(\Omega)$, $H^s(\Omega)$, $H^s_0(\Omega)$ are used and we let $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{H}^s(\Omega) = (H^s(\Omega))^d$, $\mathbf{H}^s_0(\Omega) = (H^s_0(\Omega))^d$. Negative ordered Sobolev spaces $\mathbf{H}^{-s}(\Omega)(s > 0)$ are defined as the dual space, i.e., $\mathbf{H}^{-s}(\Omega) = \{\mathbf{H}^s_0(\Omega)\}'$. We denote by $\langle \cdot \mid \cdot \rangle$ and $\|\cdot\| = \|\cdot\|_{\mathbf{L}^2(\Omega)}$, the scalar product and norm in $\mathbf{L}^2(\Omega)$, respectively. Moreover, if $\mathbf{u} \in \mathbf{L}^2(\Omega)$ is such that $\nabla \cdot \mathbf{u} \in L^2(\Omega)$, then we denote the normal trace of $\mathbf{u}$ in $\mathbf{H}^{-\frac{1}{2}}(\Gamma)$ by $\mathbf{u} \cdot \mathbf{n}$, where $\mathbf{n}$ denotes the unit outer normal vector to Γ.

We consider a stationary motion of an incompressible fluid described by the velocity and pressure $(\mathbf{v}_s, q_s)$, which is the solution to the stationary Navier-Stokes equations

$$\begin{cases} -\nu\Delta\mathbf{v}_s + (\mathbf{v}_s.\nabla)\mathbf{v}_s + \nabla q_s = \mathbf{f}_s & \text{in } \Omega, \\ \nabla \cdot \mathbf{v}_s = 0 & \text{in } \Omega, \\ \mathbf{v}_s = \mathbf{v}_b & \text{on } \Gamma_b, \\ \mathbf{v}_s = 0 & \text{on } \Gamma_l. \end{cases} \tag{1.1}$$

In this setting, $\nu > 0$ is the viscosity, $\mathbf{f}_s$ is a function in $\mathbf{L}^2(\Omega)$, $\mathbf{v}_b$ belongs to $V^{\frac{1}{2}}(\Gamma)$ defined as $V^{\frac{1}{2}}(\Gamma) = \left\{\mathbf{u} \in \mathbf{H}^{1/2}(\Gamma) : \int_\Gamma \mathbf{u} \cdot \mathbf{n}\, d\zeta = 0\right\}$. Recall [17] that a solution $(\mathbf{v}_s, q_s)$ to (1.1) is known to exist in $\mathbf{H}^1(\Omega) \times L^2_0(\Omega)$. For $T > 0$ fixed, let $Q = [0, T[\times\Omega$, $\Sigma_l = [0, T[\times\Gamma_l$ and $\Sigma_b = [0, T[\times\Gamma_b$ and consider $(\mathbf{u}, q)$ solution of the non stationary Navier-Stokes equations

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla q = \mathbf{f}_s & \text{in } Q, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } Q, \\ \mathbf{u} = \mathbf{v}_b + \mathbf{u}_b & \text{on } \Sigma_b, \\ \mathbf{u} = 0 & \text{on } \Sigma_l, \\ \mathbf{u}_0(\mathbf{x}) = \mathbf{v}_s(\mathbf{x}) + \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega. \end{cases} \tag{1.2}$$

Consequently, the couple $(\mathbf{v} = \mathbf{u} - \mathbf{v}_s, p = q - q_s)$ satisfies the following system

$$\begin{cases} (a) & \dfrac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}_s + (\mathbf{v}_s \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0 & \text{in } Q, \\ (b) & \nabla \cdot \ \mathbf{v} = 0 & \text{in } Q, \\ (c) & \mathbf{v} = \mathbf{u}_b & \text{on } \Sigma_b, \\ (d) & \mathbf{v} = 0 & \text{on } \Sigma_l, \\ (e) & \mathbf{v}(t = 0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega. \end{cases} \tag{1.3}$$

In order to stabilize the unsteady solution $\mathbf{u}$ of (1.2), for a prescribed rate of decrease $\sigma > 0$, we need to find a control $\mathbf{u}_b$ such that the components $\mathbf{v}$ of the solution $(\mathbf{v}, \nabla p)$ to the boundary value problem (1.3) satisfies the exponential decay :

$$\|\mathbf{v}(t,\mathbf{x})\| \leq C\, e^{-\sigma t}\, \|\mathbf{v}_0(\mathbf{x})\|, \quad t \in (0,\infty), \tag{1.4}$$

for a constant $C > 0$ independent of $\mathbf{v}_0(\mathbf{x})$. It's worth noticing that, in the present paper, we let $C = 1$.

The control $\mathbf{u}_b(t)$ is called a feedback if there exists a mapping $F \;:\; \mathbf{X}(\Omega) \to \mathbf{U}(\Gamma_b)$ such that

$$\mathbf{u}_b(t) = F(\mathbf{v}(t)), \quad t \in (0,\infty), \tag{1.5}$$

and the corresponding feedback law in (1.5) is pointwise in time. However, the feedback law may be chosen in a different manner, for example as

$$\mathbf{u}_b = F_0 \mathbf{v}_0, \tag{1.6}$$

where F_0 is a mapping belonging to $\mathcal{L}(\mathbf{X}(\Omega), \mathbf{U}(\Gamma_b))$, but in that case, the feedback law problème de stabilisation pointwise in time. The spaces $\mathbf{X}(\Omega)$ and $\mathbf{U}(\Gamma_b)$ will be defined accordingly. Pointwise feedback laws are usually needed in engineering applications as they are more robust with respect to perturbations in the models.

Different approaches have been pursued in the past, which first determine a linear feedback law by solving a linear control problem for the linearized system of equations (for example the Oseen system) and then use this linear feedback law in order to stabilize the original non linear system (for example the Navier-Stokes system). In such a framework, several significant questions have to be addressed. First, do we obtain a pointwise feedback law able to stabilize the linearized system ? Secondly, by assuming that F is a pointwise (in time) feedback law able to stabilize the linear system in $\mathbf{X}(\Omega)$, does F also stabilize the nonlinear system for $\mathbf{v}_0(\mathbf{x})$ in a subspace of $\{\mathbf{u} \in \mathbf{L}^2(\Omega) \;:\; \nabla \cdot \mathbf{u} = 0\}$, with $\|\mathbf{v}_0(\mathbf{x})\|$ small enough ? Finally, assuming that the existence of a feedback law stabilizing the linear system is proved, is it possible to obtain a well posed equation characterizing F, for example a Riccati equation, which can be numerically solved by classical methods ?

These questions of stabilizing the Navier-Stokes equations with a boundary control have been first addressed by A.V. Fursikov in [14, 15], where stability results for the two and three-dimensional Navier-Stokes equations are proved by employing an extension operator. With an adequate extension procedure for the initial velocity condition $\mathbf{v}_0(\mathbf{x})$ in (1.3), which requires the knowledge of the eigenfunctions and the eigenvalues of the Oseen operator, the author obtains a boundary control of the form $\mathbf{u}_b = F_0\mathbf{v}_0$, where

$F_0 \in \mathcal{L}(\mathbf{X}(\Omega), L^2([0,\infty[; \mathbf{U}(\Gamma_b)))$ and

$$\begin{aligned}
\mathbf{X}(\Omega) &= \big\{\mathbf{u} \in \mathbf{H}^{k-1}(\Omega) \ : \ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, \ \mathbf{u} = 0 \text{ on } \Gamma_l, \int_{\Gamma_b} \mathbf{u} \cdot \mathbf{n} \, d\zeta = 0\big\}, \\
\mathbf{U}(\Gamma_b) &= \big\{\mathbf{u} \in \mathbf{H}^{k-1/2}(\Gamma) : \mathbf{u} = 0 \text{ on } \Gamma_l, \int_{\Gamma_b} \mathbf{u} \cdot \mathbf{n} \, d\zeta = 0\big\},
\end{aligned}$$

with $k \geq 1$. However, if the feedback controls are well characterized, the corresponding laws are not pointwise in time.

In [24], as far as the two-dimensional case is concerned, J.-P. Raymond has obtained boundary feedback control laws, pointwise in time, where the feedback controller is determined by solving an algebraic Riccati equation obtained via the solution of an optimal control problem with

$$\begin{aligned}
\mathbf{X}(\Omega) &= \big\{\mathbf{u} \in \mathbf{H}^{1/2-\varepsilon}(\Omega) \ : \ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, \ \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma\big\}, \\
\mathbf{U}(\Gamma) &= \big\{m \, \mathbf{u} \in \mathbf{L}^2(\Gamma) : \int_{\Gamma} m \, \mathbf{u} \cdot \mathbf{n} \, d\zeta = 0\big\},
\end{aligned}$$

where $0 < \varepsilon < 1/4$ and $m \in C^2(\Gamma)$. Unfortunately, the three-dimensional case is more demanding in terms of velocity regularity, as explained in [23], and it cannot be treated in the same manner as the two-dimensional case. Indeed, in the three-dimensional case the feedback controller needs to satisfy $F(\mathbf{v})$ belonging to $\mathbf{H}^{1/4+\varepsilon/2}(0,\infty; \mathbf{L}^2(\Gamma))$ with $1/2 \leq \varepsilon$, and in the particular case $1/2 < \varepsilon$, the space $\mathbf{H}^{1/4+\varepsilon/2}([0,\infty[; \mathbf{L}^2(\Gamma))$ is a subspace of $C([0,\infty[; \mathbf{L}^2(\Gamma))$, implying that the velocity v has to satisfy the initial compatibility condition $\mathbf{v}_0|_\Gamma = F(\mathbf{v}_0)$. This is the reason why the feedback law used in [24] cannot be employed in the three-dimensional case, and why this difficulty has been overcome in [23] by introducing a time dependent feedback law in an initial transitory time interval. In order to obtain a stabilization result via the Riccati approach, particular spaces of initial conditions have to be employed that are given in [3].

The study, performed in [23], also improves in some way the results obtained in [8, 9], where a tangential boundary stabilization of two and three-dimensional Navier-Stokes equations is employed with both Riccati-based and spectral-based (tangential) feedback controllers. In [9], for the three-dimensional case which is highly demanding in terms of velocity regularity, the existence of boundary feedback laws, pointwise in time, is established by solving an optimal control problem with a cost functional involving the $L^2(0,\infty; \mathbf{H}^{3/2+\varepsilon}(\Omega))$ norm of the velocity field, for some $0 < \varepsilon$ small enough. However, such a feedback law cannot be characterized by a well posed Riccati equation, as shown in [9], and the numerical calculation of the feedback control thus becomes problematic. In [23], for the three-dimensional Navier-Stokes system, J.-P. Raymond chooses a functional involving a very weak norm of the state variable which leads to a well posed

Riccati equation.

Recall in [23], a time dependent feedback law in an initial transitory time interval was introduced. As mentioned in [2], the problem of finding a time independent feedback controller satisfying $\mathbf{v}_0|_\Gamma = F(\mathbf{v}_0)$, for a sufficiently large class of initial conditions $\mathbf{v}_0$, is not obvious. This problem has been examined in [2] for the two and three-dimensional case, and it has led to search for solutions $\mathbf{u}_b$ satisfying an extended system composed of the evolution system

$$\frac{\partial \mathbf{u}_b}{\partial t} - \Delta_B \mathbf{u}_b - \sigma\,\mathbf{n} = F(\mathbf{v}, \mathbf{u}_b), \quad \mathbf{u}_b(0) = \mathbf{v}_0|_\Gamma,$$

coupled with the original Navier-Stokes equations, where the feedback controller F now acts on the pair $(\mathbf{v}, \mathbf{u}_b)$ and Δ_B is the vector-valued Laplace Beltrami operator. The space $\mathbf{X}(\Omega)$ is now defined as

$$\mathbf{X}(\Omega) = \big\{\mathbf{u} \in \mathbf{H}^s(\Omega) \;:\; \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega,\; \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma\big\},$$

with $s \in [\frac{d-2}{2}, 1]\backslash\{1/2\}$, the oprerator F is found from a well-posed Riccati equation and the controller $\mathbf{u}_b$, localized on an arbitrary small part of Γ, can be obtained.

In the purpose of stabilizing the Navier-Stokes equations around a stationary state, the feedback control laws are determined by solving a Riccati equation in most of the studies cited above [2, 3, 7, 8, 9, 23, 24], except in the Fursikov's papers [14, 15]. The Riccati equation is obtained via the solution of an optimal control problem and it is stated in a space of infinite dimension. Although our study is only concerned with the construction of boundary controllers, the Riccati approach described above, stated in a space of infinite dimension, applies as well to the case of internal control [5, 11].

In the case the feedback controller lies in an infinite-dimensional space, an optimal control problem has to be solved, involving the minimization of an objective functional. In practice, the control is calculated through approximation via the solution of an algebraic Riccati equation, which is computationally expensive. Consequently, the use of finite-dimensional controllers may be more appropriate to stabilize the Navier-Stokes equations. Such an approach is performed in [10], in the case of an internal control, and in [1, 7, 8, 9, 22], in the case of a boundary control. Recall the Riccati equation is stated in a space of infinite dimension in [7, 8, 9]. In [1, 10, 22], the authors search for a boundary control $\mathbf{u}_b$ of finite dimension of the form

$$\mathbf{u}_b = \sum_{j=1}^{N} u_j(t)\boldsymbol{\varphi}_j(\mathbf{x}), \qquad t \geq 0,\; \mathbf{x} \in \Gamma, \tag{1.7}$$

where $(\boldsymbol{\varphi}_j)_{j=1,2,3,\ldots,N}$ is a finite-dimensional basis obtained from the eigenfunctions of some operator and $\bar{\mathbf{u}} = (u_1, u_2, u_3, \ldots, u_N)$ is a control function expressed with a feedback formulation. In [22], where $d = 2$, the feedback control is obtained from the solution of a finite-dimensional Riccati equation stated in $\mathbb{R}^{n_c \times n_c}$, where n_c is the dimension of the unstable space of the Oseen operator. The same approach is then extended in [1] for the three-dimensional case. However, in [10, 22] the minimal value of N is a priori unknown while in [1], N is greater or equal to the maximum of the geometric multiplicities of the unstable modes of the Oseen operator. Finally, finite-dimensional stabilizing feedback laws of the form of (1.7) are obtained in [6] and [4], in the case of internal and boundary control, respectively. Instead of employing the Riccati approach, a stochastic-based stabilization technique is employed in [6] which avoids the difficult computation problems related to infinite-dimensional Riccati equations. The procedure employed in [4] ressembles the form of stabilizing noise controllers designed in [6].

In all the above-mentioned studies, a linear feedback law is first determined by solving a linear control problem for the linearized system of equations and then this linear feedback is used in order to stabilize the original non linear system. However, such a procedure imposes to choose the initial velocity small enough. Further, the employed methods (e.g. the Riccati approach) require to search for the control $\mathbf{u}_b$ and the initial condition in sufficiently regular spaces, depending on whether $d = 2$ or $d = 3$. For example, in [4, Theorem 2.3], we have

$$
\begin{aligned}
\widetilde{\mathbf{H}}(\Omega) &= \left\{\mathbf{u} \in \mathbf{L}^2(\Omega)\ : \nabla \cdot \mathbf{u} = 0, \ \ \mathbf{u} \cdot \mathbf{n} = 0 \ \text{ sur } \Gamma\right\}, && (1.8)\\
\mathbf{X}(\Omega) &= \mathbf{H}^{1/2-\epsilon}(\Omega) \cap \widetilde{\mathbf{H}}(\Omega), && (1.9)
\end{aligned}
$$

in the case $d = 2$ and, for $\mathbf{v}_0 \in \mathbf{X}(\Omega)$, with $\|\mathbf{v}_0\|_{\mathbf{X}(\Omega)} < \rho$ and ρ sufficiently small, the function $\mathbf{v}$ satisfies the following stability estimate $\|\mathbf{v}\|_{\mathbf{X}(\Omega)} \leq Ce^{-\sigma t}\|\mathbf{v}_0\|_{\mathbf{X}(\Omega)}$, for all $t \geq 0$ and for some $\sigma > 0$, but the value of C is not precisely given. Note that, in the case $d = 3$, no control is proposed in [4] to stabilize the non linear Navier-Stokes equations. Further, in [1, Theorem 2], we have $\mathbf{v}_0 \in \mathbf{H}^s(\Omega)$ with $\nabla \cdot \mathbf{v}_0 = 0$, $s \in [0, 1/2)$ and $\|P\mathbf{v}_0\|_{\mathbf{H}^s(\Omega)} \leq c$ in the case $d = 2$, where P is the Leray projector, and $\mathbf{v}_0 \in \mathbf{H}^s_0(\Omega)$ with $\nabla \cdot \mathbf{v}_0 = 0$, $\bar{\mathbf{u}} = 0$, $s \in (1/2, 1]$ and $\|\mathbf{v}_0\|_{\mathbf{H}^s_0(\Omega)} \leq c$ in the case $d = 3$, and stability estimates are also obtained.

In this paper, a new approach is proposed. Instead of obtaining the feedback law by first solving a linear control problem for the linearized system of equations, eventually via the resolution of a Riccati equation, an extended system is considered. Indeed, in (1.3) the boundary control $\mathbf{u}_b$ is rewritten on the form $\mathbf{u}_b = \alpha(t)\mathbf{g}(\mathbf{x})$ on Σ_b, where $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$ is assumed to verify $\mathbf{g} = 0$ on Γ_l, $\mathbf{g} \cdot \mathbf{n} \neq 0$ on Γ_b and $\int_{\Gamma_b} \mathbf{g} \cdot \mathbf{n} \, d\zeta = 0$. The quantity $\alpha(t)$ is a priori unknown. In order to stabilize (1.3), with $\mathbf{u}_b = \alpha(t)\mathbf{g}(\mathbf{x})$ on Σ_b, by employing energy a priori estimation technics, the quantity $\alpha(t)$ is found to satisfy the relation

$$f(\mathbf{v},\alpha) = \int_{\Gamma_b} [\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}] \cdot \mathbf{g} \, d\zeta, \tag{1.10}$$

where f is a polynomial in $\alpha(t)$ of degree 2. Note that $\alpha(t)$ depends nonlinearly on v and hence $\alpha(t)$, which reads $\alpha(\mathbf{v}(t))$, satisfies a nonlinear feedback law. Such a feedback, pointwise in time, ressembles to (1.5) but the mapping F is nonlinear here.

The system (1.3) is then extended by adding (1.10), and the extended system, namely (1.3) and (1.10) with $\mathbf{u}_b = \alpha(t)\mathbf{g}(\mathbf{x})$ on Σ_b, is then solved in order to determined $\alpha(t)$, leading to the determination of the boundary control $\mathbf{u}_b$. Such a boundary representation of $\mathbf{u}_b$ is also employed in [21] in the two-dimensional case, where a linear feedback control $d\alpha(t)/dt$ is obtained via the solution of a Riccati equation stated in a space of infinite dimension. In the present paper, however, the quantities $\alpha(t)$, and hence $\mathbf{u}_b$, are computed at the discrete level. Further, contrary to (1.7) and [21], where $u_j(t)$, $j = 1, 2, 3, \ldots, N$, and $d\alpha(t)/dt$, respectively, are linear feedbacks, $\alpha(t)$ is nonlinear here and it is thus calculated through a Galerkin procedure instead of being the solution of a finite-dimensional Riccati equation, for example.

Note that the Galerkin procedure first consists of building a sequence of approximated solutions via an adequate Galerkin basis. Because the energy bounds are not sufficient to pass to the limit in the weak formulation, additional bounds are obtained. A compactness result then permits to pass to the limit in the system satisfied by the approximated solution, leading to the existence of at least one weak solution. Such a procedure relies on technics previously introduced in [19], but it is worth to note that the work performed in [19] is not related to a stabilization problem.

The approach proposed in this paper has several advantages. First, the stabilization result in (1.4), i.e. $\|\mathbf{v}(t,\mathbf{x})\| \leq C\, e^{-\sigma t}\, \|\mathbf{v}_0(\mathbf{x})\|$, for $t \in (0,\infty)$, is obtained with $C = 1$ and for an arbitrary initial data $\mathbf{v}_0$ belonging to $\mathbf{H}(\Omega) = \left\{\mathbf{u} \in \mathbf{L}^2(\Omega)\ : \nabla \cdot \mathbf{u} = 0,\ \ \mathbf{u} \cdot \mathbf{n} = 0 \ \text{ sur } \Gamma_l\right\}$, implying less regularity on $\mathbf{v}_0$ than in the case of the previous studies cited above, for example see (1.9). Further, the regularity results are independent of d and they are thus obtained in the two and three-dimensional case as well.

The paper is organized as follows. In section 2, the notations and mathematical preliminaries are introduced. The stabilization problem is formulated in Section 3, and the existence of the solution of the nonlinear Navier-Stokes system is established and the existence analysis is carried out by applying the Galerkin method. Finally, some concluding remarks complete the study in Section 4.

2 Notation and Preliminaries

2.1 Function Spaces

Several spaces of free divergence functions are now introduced :

$$\boldsymbol{\mathcal{V}}(\Omega) = \{\mathbf{u} \in \boldsymbol{\mathcal{D}}(\Omega) \;:\; \nabla\cdot\mathbf{u} = 0\}, \tag{1.11}$$

$$\mathbf{V}(\Omega) = \Big\{\mathbf{u} \in \mathbf{H}^1(\Omega) \;:\; \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega,\; \mathbf{u} = 0 \text{ on } \Gamma_l, \int_{\Gamma_b} \mathbf{u}\cdot\mathbf{n}\, d\zeta = 0\Big\}, \tag{1.12}$$

$$\mathbf{V}_0(\Omega) = \{\mathbf{u} \in \mathbf{H}^1_0(\Omega) \;:\; \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega\}, \tag{1.13}$$

$$\mathbf{H}(\Omega) = \Big\{\mathbf{u} \in \mathbf{L}^2(\Omega) \;: \nabla\cdot\mathbf{u} = 0,\;\; \mathbf{u}\cdot\mathbf{n} = 0 \;\text{ on } \Gamma_l, \int_{\Gamma_b} \mathbf{u}\cdot\mathbf{n}\, d\zeta = 0\Big\}. \tag{1.14}$$

Because $\mathbf{V}(\Omega)$ is a closed subspace of $\mathbf{H}^1(\Omega)$, we have, by definition $\|\cdot\|_{\mathbf{V}(\Omega)} = \|\cdot\|_{\mathbf{H}^1(\Omega)}$.

Definition 2.1. *Let $\mathbf{V}^{1/2}(\Gamma_b)$ be the space of trace functions that, if extended by zero over Γ, belongs to $\mathbf{H}^{1/2}(\Gamma)$.*

Let g such that $\mathbf{g} \in \mathbf{V}^{1/2}(\Gamma_b)$ with $\mathbf{g}\cdot\mathbf{n} \neq 0$ on Γ_b and $\int_{\Gamma_b} \mathbf{g}\cdot\mathbf{n}\, d\zeta = 0$, the solution of (1.3) coupled with (1.10) is searched in

$$W(Q) = \{(\mathbf{v}, \alpha) \in \mathbf{V}(\Omega) \times \mathbb{R},\; \text{s.t. } \mathbf{v} = \alpha\mathbf{g} \text{ on } \Gamma_b\}. \tag{1.15}$$

The following lemma [19], will be used in the sequel.

Lemma 2.2. *There exists a constant $C_b > 0$ such that, for all $(\mathbf{v}, \alpha) \in W(Q)$, we have*

$$|\alpha| \leq C_b \|\mathbf{v}\|. \tag{1.16}$$

We now define an Galerkin basis for the space $W(Q)$.

2.2 A Galerkin basis for the space $W(Q)$

Let $\{\mathbf{z}_j, \lambda_j, j = 1, 2, 3, \cdots\}$ be the eigenfunctions and eigenvalues of the following spectral problem for the Stokes operator :

$$-\Delta\mathbf{z}_j + \nabla p_j = \lambda_j \mathbf{z}_j, \qquad \nabla\cdot\mathbf{z}_j = 0 \quad \text{in } \Omega; \qquad \mathbf{z}_j|_\Gamma = 0. \tag{1.17}$$

As shown in [25], $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty$ as $j \to \infty$, and $\{\mathbf{z}_j\}$ forms an orthonormal basis in $\mathbf{V}_0(\Omega)$ verifying :

$$\begin{cases} \langle \mathbf{z}_j, \mathbf{z}_k \rangle = \delta_{jk}, \\ \langle \nabla \mathbf{z}_j, \nabla \mathbf{z}_k \rangle = \lambda_j \delta_{jk}, \quad \forall j, k = 1, 2, 3, \ldots \end{cases} \tag{1.18}$$

The space $W(Q)$, defined in (1.15), is then rewritten as

$$W(Q) = \text{span}(\mathbf{z}_n)_{\{n \in \mathbb{N}^*\}} \oplus \text{span}(\mathbf{w}), \tag{1.19}$$

where w satisfies the following system

$$-\nu \Delta \mathbf{w} + \nabla q = 0, \quad \nabla \cdot \mathbf{w} = 0 \text{ in } \Omega, \quad \mathbf{w} = 0 \text{ on } \Gamma_l, \quad \mathbf{w} = \mathbf{g} \text{ on } \Gamma_b. \tag{1.20}$$

Since g satisfy $\int_{\Gamma_b} \mathbf{g} \cdot \mathbf{n} \, d\zeta = 0$, system (1.20) hence admits a unique solution $(\mathbf{w}, q) \in \mathbf{V}(\Omega) \times L^2_0(\Omega)$, where $L^2_0(\Omega)$ is the pressure space with zero mean value :

$$L^2_0(\Omega) = \left\{ p \in L^2(\Omega), \int_\Omega p(\mathbf{x}) \, d\mathbf{x} = 0 \right\}.$$

Note that the existence and uniqueness of $(\mathbf{w}, q)$ in (1.20) can be deduced from [25].

2.3 Linear Forms

In order to define a weak form of the Navier-Stokes equations, we introduce the continuous bilinear forms

$$a(\mathbf{v}_1, \mathbf{v}_2) = \int_\Omega \nabla \mathbf{v}_1 : \nabla \mathbf{v}_2 \, d\mathbf{x}, \quad \forall (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega),$$

and the trilinear form :

$$b(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \int_\Omega (\mathbf{v}_1 \nabla) \mathbf{v}_2 \cdot \mathbf{v}_3 \, d\mathbf{x}, \quad \forall (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega).$$

By integration by parts, the following properties hold true

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = \frac{\alpha^2}{2} \int_{\Gamma_b} |\mathbf{g}|^2 (\mathbf{u} \cdot \mathbf{n}) \, d\zeta, \ \forall \mathbf{u} \in \mathbf{V}(\Omega), \ \forall (\mathbf{v}, \alpha) \in W(Q), \tag{1.21}$$

$$b(\mathbf{v}, \mathbf{v}, \mathbf{v}) = \frac{\alpha^3}{2} \int_{\Gamma_b} |\mathbf{g}|^2 (\mathbf{g} \cdot \mathbf{n}) \, d\zeta, \ \forall (\mathbf{v}, \alpha) \in W(Q). \tag{1.22}$$

Thanks to Hölder inequality, we obtain

$$|b(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)| \leq \|\mathbf{v}_1\|_{\mathbf{L}^2(\Omega)} \|\nabla \mathbf{v}_2\|_\infty \|\mathbf{v}_3\|_{\mathbf{L}^2(\Omega)}, \quad \forall \mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3 \in \mathbf{H}^1(\Omega), \tag{1.23}$$

where $\|\cdot\|_\infty = \|\cdot\|_{\mathbf{L}^\infty(\Omega)}$.

3 Stability Result

3.1 The stabilization Problem

In order to stabilize the non stationary Navier-Stokes System (1.3), we choose to search the solution $\mathbf{v}$ in the form $\mathbf{v} = \mathbf{z} + \alpha \mathbf{w}$, where $\mathbf{z} \in \mathbf{V}_0(\Omega)$, and α and $\mathbf{w}$ satisfy (1.10) and (1.20), respectively. We then have $\mathbf{v} = \alpha \mathbf{g}$ on Γ_b as $\mathbf{z} = \mathbf{0}$ on Γ. Consequently, the state $(\mathbf{v}, p)$ satisfies the following extended coupled system :

$$\begin{cases} (a) & \dfrac{\partial \mathbf{v}}{\partial t} - \nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}_s + (\mathbf{v}_s \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0 & in\ Q, \\ (b) & \nabla \cdot\ \mathbf{v} = 0 & in\ Q, \\ (c) & \mathbf{v} = \alpha(t)\mathbf{g}(\mathbf{x}) & on\ \Sigma_b, \\ (d) & \mathbf{v} = 0 & on\ \Sigma_l, \\ (e) & \mathbf{v}(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & in\ \Omega, \\ (f) & \displaystyle\int_{\Gamma_b} [\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}] \cdot \mathbf{g}\ d\zeta = f(\mathbf{v}, \alpha), & \end{cases} \tag{1.24}$$

where

$$f(\mathbf{v}, \alpha)(t) = a\alpha^2(t) + b\alpha(t) - \sigma_0 \|\mathbf{v}(t)\|^2 \alpha(t) - \nu\lambda_1 \left(\|\mathbf{w}\|^2 \alpha(t) + 2\langle \mathbf{w}, \mathbf{z}(t) \rangle \right). \tag{1.25}$$

with $\sigma_0 > 0$ is a constant, λ_1 is the smallest positive eigenvalue of (1.17) and

$$a = \frac{1}{2} \int_{\Gamma_b} |\mathbf{g}|^2 (\mathbf{g} \cdot \mathbf{n})\ d\zeta \quad \text{and} \quad b = \frac{1}{2} \int_{\Gamma_b} |\mathbf{g}|^2 (\mathbf{v}_s \cdot \mathbf{n})\ d\zeta.$$

Recall that α is a priori unknown and thanks to (1.24-f), it satisfies a nonlinear feedback law leading to search for $\alpha(\mathbf{v}(t))$. Because (1.24-f) is independent of $\mathbf{x}$, $\alpha(\mathbf{v}(t))$ is a function of t only. For the sake of simplicity, $\alpha(\mathbf{v}(t))$ is written α in the sequel.

3.2 The variational formulation

We first state to consider the variational formulation of the extended Navier-Stokes system.

Definition 3.1. *Let $T > 0$ be an arbitrary number, we shall say that $(\mathbf{v}, \alpha)$ is a weak solution of (1.24) on $[0, T)$ if*

- $\mathbf{v} \in [L^\infty(0, T; \mathbf{H}(\Omega)) \cap L^2(0, T; \mathbf{V}(\Omega))]$,
- $\exists \alpha \in L^\infty(0, T)$ *such that* $\mathbf{v} = \alpha \mathbf{g}$ *on* Γ_b,

$$\begin{cases} (a) \quad \langle d_t \mathbf{v}, \widetilde{\mathbf{v}} \rangle + \nu a(\mathbf{v}, \widetilde{\mathbf{v}}) + b(\mathbf{v}, \mathbf{v}_s, \widetilde{\mathbf{v}}) + b(\mathbf{v}_s, \mathbf{v}, \widetilde{\mathbf{v}}) + b(\mathbf{v}, \mathbf{v}, \widetilde{\mathbf{v}}) = \widetilde{\alpha} f(\mathbf{v}, \alpha), \\ (b) \quad \mathbf{v}(0) = \mathbf{v}_0, \end{cases} \tag{1.26}$$

for all $(\widetilde{\mathbf{v}}, \widetilde{\alpha}) \in W(Q)$.

Theorem 3.2. *Let λ_1 the smallest positive eigenvalue of (1.17), and assume that the steady state $\mathbf{v}_s$, the initial condition $\mathbf{v}_0$ and the profile $\mathbf{g}$ satisfy*

$$\bar{\sigma} = \nu \lambda_1 - \|\nabla \mathbf{v}_s\|_\infty > 0, \tag{1.27}$$

$$\mathbf{v}_0 \in \mathbf{H}(\Omega), \quad (\mathbf{v}_0 \cdot \mathbf{n})\, \mathbf{n} \in \mathbf{H}^{1/2}(\Gamma_b), \tag{1.28}$$

$$\mathbf{g} \in \mathbf{V}^{1/2}(\Gamma_b) \quad \textit{and} \quad \alpha_0\, \mathbf{g} \cdot \mathbf{n} = \mathbf{v}_0 \cdot \mathbf{n} \ \textit{ on } \ \Gamma_b \ \textit{with} \ \ \mathbf{g} \cdot \mathbf{n} \neq 0,\ \alpha_0 \in \mathbb{R}. \tag{1.29}$$

For arbitrary initial data $\mathbf{v}_0$ satisfying (1.28), there exists a solution $(\mathbf{v}, \alpha)$ in the sense of definition 3.1, and a distribution p on Q such that (1.24) holds. Moreover, $\mathbf{v}$ satisfies the following estimates :

$$\|\mathbf{v}(t)\| \ \leq \ \|\mathbf{v}_0\|\, e^{-\sigma(t)}, \ \forall t > 0, \tag{1.30}$$

$$\int_0^T \|\nabla \mathbf{v}(t)\|^2 dt \ \leq \ C \|\mathbf{v}_0\|^2, \tag{1.31}$$

where $C > 0$ is a constant, $\sigma(t) = \sigma_1 t + \sigma_0 \int_0^t \alpha^2(s) ds \geq 0$, and the constants σ_0 and σ_1 satisfy $\sigma_0 > 0$ and $0 < \sigma_1 \leq \bar{\sigma}$.

Note that the rate of decrease $\sigma(t)$ depends on the control α and the constant σ_0 may be regarded as an accelerator in terms of stabilization.

Remark 3.3. *With the condition (1.27), the equilibrium state $\mathbf{v}_s$ in (1.1) is naturally stable in the sense that the system (1.26) stabilizes by itself when α is identically zero. This explains why the choice of the initial perturbation $\mathbf{v}_0$, in Theorem 3.2, is arbitrary. However, as shown in Proposition 3.1, the control α is not identically zero as soon as the initial perturbation $\mathbf{v}_0$ and the profile $\mathbf{g}$ satisfy (1.28)-(1.29) with $\mathbf{v}_0 \cdot \mathbf{n} \neq 0$. The theoretical case $\mathbf{v}_0 \cdot \mathbf{n} = 0$ remains an open question.*

Proof. Let us begin with the proof of the stability estimates followed by the existence result.

3.3 A priori estimates

Taking $(\widetilde{\mathbf{v}}, \widetilde{\alpha}) = (\mathbf{v}, \alpha) \in W(Q)$ in (1.26-a) leads to

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\|\nabla\mathbf{v}\|^2 + b(\mathbf{v},\mathbf{v},\mathbf{v}) + b(\mathbf{v}_s,\mathbf{v},\mathbf{v}) + b(\mathbf{v},\mathbf{v}_s,\mathbf{v}) = \alpha f(\mathbf{v},\alpha). \tag{1.32}$$

Let us estimate the terms in the left-hand side of (1.32). According to (1.21)-(1.23), we obtain

$$\begin{aligned} b(\mathbf{v},\mathbf{v},\mathbf{v}) &= \frac{\alpha^3}{2}\int_{\Gamma_b} |\mathbf{g}|^2(\mathbf{g}\cdot\mathbf{n})d\zeta, && (1.33)\\ b(\mathbf{v}_s,\mathbf{v},\mathbf{v}) &= \frac{\alpha^2}{2}\int_{\Gamma_b} |\mathbf{g}|^2(\mathbf{v}_s\cdot\mathbf{n})d\zeta && (1.34)\\ |b(\mathbf{v},\mathbf{v}_s,\mathbf{v})| &\leq \|\nabla\mathbf{v}_s\|_\infty\|\mathbf{v}\|^2. && (1.35)\end{aligned}$$

Using (1.25) and (1.33)-(1.35) in (1.32), leads to

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\|\nabla\mathbf{v}\|^2 \leq \|\nabla\mathbf{v}_s\|_\infty\|\mathbf{v}\|^2 - \sigma_0\|\mathbf{v}\|^2\alpha^2 - \nu\lambda_1\left(\|\mathbf{w}\|^2\alpha^2 + 2\alpha\langle\mathbf{w},\mathbf{z}\rangle\right). \tag{1.36}$$

Due to (1.20), we have $\langle\nabla\mathbf{w},\nabla\mathbf{z}\rangle = 0$ and from (1.36) we deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\alpha^2\|\nabla\mathbf{w}\|^2 + \nu\|\nabla\mathbf{z}\|^2 &\leq \|\nabla\mathbf{v}_s\|_\infty\|\mathbf{v}\|^2 - \sigma_0\|\mathbf{v}\|^2\alpha^2\\ &\quad - \nu\lambda_1\left(\|\mathbf{w}\|^2\alpha^2 + 2\alpha\langle\mathbf{w},\mathbf{z}\rangle\right). \end{aligned}\tag{1.37}$$

Since

$$\lambda_1\|\mathbf{z}\|^2 = \lambda_1\sum_{i=1}^{\infty}\theta_i \leq \sum_{i=1}^{\infty}\lambda_i\theta_i = \|\nabla\mathbf{z}\|^2,$$

and using $\mathbf{v} = \mathbf{z} + \alpha\mathbf{w}$, we obtain from (1.37)

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\alpha^2\|\nabla\mathbf{w}\|^2 + \nu\lambda_1\|\mathbf{v}\|^2 \leq \|\nabla\mathbf{v}_s\|_\infty\|\mathbf{v}\|^2 - \sigma_0\|\mathbf{v}\|^2\alpha^2. \tag{1.38}$$

For all σ_1 such that $0 < \sigma_1 \leq \bar{\sigma} = \nu\lambda_1 - \|\nabla\mathbf{v}_s\|_\infty$, we have

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\alpha^2\|\nabla\mathbf{w}\|^2 + (\sigma_1 + \sigma_0\alpha^2)\|\mathbf{v}\|^2 \leq 0 \tag{1.39}$$

and omitting the second term in the left hand side of (1.39) leads to

$$\frac{d}{dt}\|\mathbf{v}\|^2 + 2(\sigma_1 + \sigma_0\alpha^2)\|\mathbf{v}\|^2 \leq 0. \tag{1.40}$$

Multiplying (1.40) by $e^{2\sigma(t)}$, where $\sigma(t) = \sigma_1 t + \sigma_0 \int_0^t \alpha^2(s)ds \geq 0$, we obtain

$$\frac{d}{dt}\left(e^{2\sigma(t)}\|\mathbf{v}\|^2\right) \leq 0$$

and consequently,

$$\|\mathbf{v}\| \leq \|\mathbf{v}_0\| e^{-\sigma(t)}. \tag{1.41}$$

By omitting the third term in the left hand side of (1.39) we deduce

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\alpha^2\|\nabla \mathbf{w}\|^2 \leq 0$$

and integrating from 0 to t yields

$$\|\mathbf{v}\|^2 + 2\nu \int_0^t \alpha^2 \|\nabla \mathbf{w}\|^2 ds \leq \|\mathbf{v}_0\|^2,$$

leading to

$$\int_0^t \alpha^2 ds \leq \frac{\|\mathbf{v}_0\|^2}{2\nu\|\nabla \mathbf{w}\|^2}. \tag{1.42}$$

Since $\mathbf{v} = \mathbf{z} + \alpha\mathbf{w}$, we substitute $\|\mathbf{w}\|^2\alpha^2 + 2\alpha\langle \mathbf{w}, \mathbf{z}\rangle = \|\mathbf{v}\|^2 - \|\mathbf{z}\|^2$ in the two last terms in the right hand side of (1.36), and this leads to

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\|\nabla \mathbf{v}\|^2 &\leq \|\nabla \mathbf{v}_s\|_\infty \|\mathbf{v}\|^2 - \nu\lambda_1(\|\mathbf{v}\|^2 - \|\mathbf{z}\|^2) = \nu\lambda_1\|\mathbf{z}\|^2 - \bar{\sigma}\|\mathbf{v}\|^2 \\
&\leq \nu\lambda_1\|\mathbf{z}\|^2 = \nu\lambda_1\|\mathbf{v} - \alpha\mathbf{w}\|^2 \\
&\leq 2\nu\lambda_1\|\mathbf{v}\|^2 + 2\nu\lambda_1\alpha^2\|\mathbf{w}\|^2.
\end{aligned} \tag{1.43}$$

Integrating (1.43) from 0 to t yields

$$\|\mathbf{v}\|^2 + 2\nu\int_0^t \|\nabla \mathbf{v}\|^2 ds \leq \|\mathbf{v}_0\|^2 + 4\nu\lambda_1 \int_0^t \|\mathbf{v}\|^2 ds + 4\nu\lambda_1\|\mathbf{w}\|^2 \int_0^t \alpha^2 ds, \tag{1.44}$$

and employing (1.41) and (1.42) we obtain

$$\|\mathbf{v}\|^2 + 2\nu\int_0^t \|\nabla \mathbf{v}\|^2 ds \leq \left(1 + 2\lambda_1\frac{\|\mathbf{w}\|^2}{\|\nabla \mathbf{w}\|^2} + 4\nu\lambda_1\int_0^t e^{-2\sigma(t)}ds\right)\|\mathbf{v}_0\|^2.$$

Because $\sigma(t) = \sigma_1 t + \sigma_0 \int_0^t \alpha^2(s)ds$, we have $\sigma(t) \geq \sigma_1 t$, and hence

$$\|\mathbf{v}\|^2 + 2\nu \int_0^t \|\nabla \mathbf{v}\|^2 ds \leq \left(1 + 2\lambda_1 \frac{\|\mathbf{w}\|^2}{\|\nabla \mathbf{w}\|^2} + \frac{2\nu\lambda_1}{\sigma_1}\left(1 - e^{-2\sigma_1 t}\right)\right) \|\mathbf{v}_0\|^2.$$

Therefore, we obtain the a priori estimate

$$\int_0^t \|\nabla \mathbf{v}\|^2 ds \leq \frac{1}{\nu}\left(\frac{1}{2} + \lambda_1 \frac{\|\mathbf{w}\|^2}{\|\nabla \mathbf{w}\|^2} + \frac{\nu\lambda_1}{\sigma_1}\right) \|\mathbf{v}_0\|^2. \tag{1.45}$$

3.4 Existence

The proof of the existence follows a standard procedure. In a first step a sequence of approximate solutions using a Galerkin method is built. A compactness result from [20] allows us to pass to the limit in the system satisfied by the approximated solutions.

3.4.1 The Galerkin Method

For all $m \in \mathbb{N}$, we define the space W_m as :

$$W_m = span(\{\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m\}),$$

where $\mathbf{w}_0 = \mathbf{w}$ and $\mathbf{w}_i = \mathbf{z}_i$, $i = 1, 2, 3, \cdots, m$. Then for $(\mathbf{v}_m, \phi_{0m}) \in W_m$, $\mathbf{v}_m = \sum_{i=0}^m \phi_{im} \mathbf{w}_i$ and we define the following finite-dimensional problem

$$\begin{cases} (a) & \langle d_t \mathbf{v}_m, \mathbf{w}_j\rangle + \nu a(\mathbf{v}_m, \mathbf{w}_j) + b(\mathbf{v}_m, \mathbf{v}_s, \mathbf{w}_j) + b(\mathbf{v}_s, \mathbf{v}_m, \mathbf{w}_j) \\ & + b(\mathbf{v}_m, \mathbf{v}_m, \mathbf{w}_j) = \delta_{0j} f(\mathbf{v}_m, \phi_{0m}), \text{ for } j = 0, 1, 2, \cdots, m, \\ (b) & \langle \mathbf{v}_m(0) - \mathbf{v}_0, \mathbf{w}_j\rangle = 0, \text{ for } j = 0, 1, 2, \cdots, m. \end{cases} \tag{1.46}$$

where δ_{ij} defined the Kronecker symbol and

$$f(\mathbf{v}_m, \phi_{0m}) = a\phi_{0m}^2 + b\phi_{0m} - \sigma_0 \|\mathbf{v}_m\|^2 \phi_{0m} - \nu\lambda_1 \left(\|\mathbf{w}\|^2 \phi_{0m} + 2\langle \mathbf{w}, \mathbf{z}_m\rangle\right), \tag{1.47}$$

with

$$\mathbf{z}_m = \sum_{i=1}^m \phi_{im} \mathbf{w}_i.$$

Lemma 3.4. *The discrete problem (1.46) has a unique solution $\mathbf{v}_m \in W^{1,\infty}(0, T; W_m)$. Moreover this solution satisfies :*

$$\|\mathbf{v}_m\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{v}_m\|_{L^2(0,T;\mathbf{H}^1(\Omega))} \leq C, \tag{1.48}$$

where C is a positive constant independent of m.

Proof. We rewrite (1.46) in terms of the unknown ϕ_{im}, $i = 0 \cdots m$, and we obtain

$$\begin{cases} \displaystyle\sum_{i=0}^{m} \frac{d\phi_{im}}{dt}\langle \mathbf{w}_i, \mathbf{w}_j\rangle + \sum_{i=0}^{m} \phi_{im}(\nu\, a(\mathbf{w}_i, \mathbf{w}_j) + b(\mathbf{v}_s, \mathbf{w}_i, \mathbf{w}_j) + b(\mathbf{w}_i, \mathbf{v}_s, \mathbf{w}_j)) \\ \qquad\qquad \displaystyle + \sum_{i,k=0}^{m} \phi_{km}\phi_{im} b(\mathbf{w}_i, \mathbf{w}_k, \mathbf{w}_j) = \delta_{0j} f(\mathbf{v}_m, \phi_{0m}), \\ \displaystyle\sum_{i=0}^{m} \phi_{im}(0)\langle \mathbf{w}_i, \mathbf{w}_j\rangle = \langle \mathbf{v}_0, \mathbf{w}_j\rangle. \end{cases} \tag{1.49}$$

Because the matrix with elements $\langle \mathbf{w}_i, \mathbf{w}_j\rangle$ $(0 \le i, j \le m)$ is nonsingular, (1.49) reduces to a nonlinear system with constant coefficients

$$\begin{cases} \displaystyle\frac{d\phi_{im}}{dt} + \sum_{j=0}^{m} \phi_{jm} X_{ij} + \sum_{j,k=0}^{m} \phi_{km}\phi_{jm} Y_{ijk} = f(\mathbf{v}_m, \phi_{0m}) \sum_{j=0}^{m} \delta_{0j} Z_{ij}, \\ \displaystyle\phi_{im}(0) = \sum_{j=0}^{m} \langle \mathbf{v}_0, \mathbf{w}_j\rangle Z_{ij}, \end{cases} \tag{1.50}$$

where $X_{ij}, Y_{ijk}, Z_{ij}, \in \mathbb{R}$. Then, there exists T_m $(0 < T_m \le T)$ such that the nonlinear differential system (1.50) has a maximal solution defined on some interval $[0, T_m]$. In order to show that T_m is independent of m, it is sufficient to verify the boundedness of ϕ_{im}, and hence the boundedness of the L^2-norm of $\mathbf{v}_m$ independently of m. Following the same procedure as for the derivation of the a priori estimates (1.41) and (1.45), yields

$$\begin{cases} (a) \quad \|\mathbf{v}_m\|^2 \le \|\mathbf{v}_0\|^2 \, e^{-2\sigma(t)}, \\ (b) \quad \displaystyle\int_0^T \|\nabla \mathbf{v}_m\|^2 dt \le C\|\mathbf{v}_0\|^2. \end{cases} \tag{1.51}$$

Consequently, according to (1.51-a), we obtain $T_m = T$. $\square$

Moreover, a consequence of the a priori estimates (1.51) is that $(\mathbf{v}_m)_m$ is bounded in $L^2(0, T; \mathbf{V}(\Omega))$ and $L^\infty(0, T; \mathbf{H}(\Omega))$. Therefore, for a subsequence of $\mathbf{v}_m$ (still denoted by $\mathbf{v}_m$), the estimates in (1.51) yield the following weak convergences as m tends to ∞ :

$$\begin{cases} \mathbf{v}_m \rightharpoonup \mathbf{v} \text{ weakly in } L^2(0, T; \mathbf{V}(\Omega)), \\ \mathbf{v}_m \rightharpoonup \mathbf{v} \text{ weakly* in } L^\infty(0, T; \mathbf{H}(\Omega)). \end{cases} \tag{1.52}$$

Nevertheless, the convergences in (1.52) are not sufficient to pass to the limit in the weak formulation (1.46), because of the presence of the convection term. Consequently, we need to obtain additional bounds in order to utilize the compactness theory on the sequence of approximated solution $(\mathbf{v}_m)_m$.

3.4.2 Additional bounds

As in [20], let us assume that $B_0,\ B$ and B_1 are three Hilbert spaces such that $B_0 \subset B \subset B_1$. If $\mathbf{v} : \mathbb{R} \to B_1$ is a function, we denote by $\widehat{\mathbf{v}}$ its Fourier transform

$$\widehat{\mathbf{v}}(\tau) = \int_{-\infty}^{+\infty} e^{-2i\pi t\tau}\mathbf{v}(t)dt.$$

Let us recall the following identity about the Fourier transform of differential operators :

$$\widehat{D_t^\gamma \mathbf{v}(\tau)} = (2i\pi\tau)^\gamma \widehat{\mathbf{v}}(\tau),$$

for a given $\gamma > 0$, and let us define the space

$$H^\gamma(\mathbb{R}; B_0, B_1) = \{\mathbf{u} \in L^2(\mathbb{R}, B_0), D_t^\gamma \mathbf{u} \in L^2(\mathbb{R}, B_1)\}.$$

The space $H^\gamma(\mathbb{R}; B_0, B_1)$ is endowed with the norm

$$\|\mathbf{v}\|_{H^\gamma(\mathbb{R};B_0,B_1)} = (\|\mathbf{v}\|^2_{L^2(\mathbb{R};B_0)} + \||\tau|^\gamma \widehat{\mathbf{v}}\|^2_{L^2(\mathbb{R};B_1)})^{\frac{1}{2}}.$$

We also define $H^\gamma(0, T; B_0, B_1)$, as the space of functions obtained by restriction to $[0, T]$ of functions of $H^\gamma(\mathbb{R}; B_0, B_1)$. Further, we recall the following result [20] :

Lemma 3.5. *Let $B_0,\ B$ and B_1 be three Hilbert spaces such that $B_0 \subset B \subset B_1$ and B_0 is compactly embedded in B. Then for all $\gamma > 0$, the injection $H^\gamma(0, T; B_0, B_1) \to L^2(0, T; B)$ is compact.*

For small enough ε, this lemma is used later with

$$B_0 = \mathbf{V}(\Omega), \qquad B = \mathbf{H}(\Omega), \qquad B_1 = \mathbf{H}(\Omega), \qquad \gamma = \frac{1}{4} - \epsilon.$$

The main result of the present section, based on utilizing Lemma 3.5, is furnished by the following lemma :

Lemma 3.6. *The sequence $\mathbf{v}_m$ is bounded in $H^\gamma(0, T; \mathbf{V}(\Omega), \mathbf{H}(\Omega))$ for $0 \le \gamma \le \frac{1}{4} - \epsilon$.*

Proof. We denote by $\bar{\mathbf{v}}_m$ the extension of $\mathbf{v}_m$ by zero 0 for $t < 0$ and $t > T$, and $\widehat{\mathbf{v}}_m$ the Fourier transform with respect to time of $\bar{\mathbf{v}}_m$. It is classical that since $\bar{\mathbf{v}}_m$ has two discontinuities at 0 and T, in the distributional sense, the derivative of $\bar{\mathbf{v}}_m$ is given by

$$\frac{d}{dt}\bar{\mathbf{v}}_m = \bar{\mathbf{u}}_m + \mathbf{v}_m(0)\delta_0 - \mathbf{v}_m(T)\delta_T, \tag{1.53}$$

where $\delta_0,\ \delta_T$ are Dirac distributions at 0 and T, and

$$\bar{\mathbf{u}}_m = \mathbf{v}_m^{'} = \text{the derivative of } \mathbf{v}_m \text{ on } [0,T].$$

After a Fourier transformation, (1.53) gives

$$2i\pi\tau\widehat{\mathbf{v}}_m(\tau) = \widehat{\mathbf{u}}_m(\tau) + \mathbf{v}_m(0) - \mathbf{v}_m(T)e^{-2i\pi\tau T},$$

where $\widehat{\mathbf{v}}_m$ and $\widehat{\mathbf{u}}_m$ denote the Fourier transforms of $\bar{\mathbf{v}}_m$ and $\bar{\mathbf{u}}_m$ respectively. Since we already know that $\mathbf{v}_m$ is uniformly bounded in $L^2(0,T,\mathbf{V}(\Omega))$, it remains to prove that

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma}\|\widehat{\mathbf{v}}_m(\tau)\|d\tau \leq C. \tag{1.54}$$

We have that $\bar{\mathbf{v}}_m$ satisfies

$$\begin{aligned}
&\int_\Omega \frac{\partial\bar{\mathbf{v}}_m}{\partial t}\cdot\widetilde{\mathbf{v}}\ d\mathbf{x} + \nu\int_\Omega \nabla\bar{\mathbf{v}}_m : \nabla\widetilde{\mathbf{v}}\ d\mathbf{x} + \int_\Omega G_m\cdot\widetilde{\mathbf{v}}\ d\mathbf{x} + \int_\Omega G_m^0\cdot\widetilde{\mathbf{v}}\ d\mathbf{x} + \int_\Omega G_m^1\cdot\widetilde{\mathbf{v}}\ d\mathbf{x} \\
&= -\int_\Omega \bar{\mathbf{v}}_m(T)\cdot\widetilde{\mathbf{v}}\delta_T\ d\mathbf{x} + \int_\Omega \bar{\mathbf{v}}_m(0)\cdot\widetilde{\mathbf{v}}\delta_0\ d\mathbf{x} + \widetilde{\alpha}H_m, \quad \forall(\widetilde{\mathbf{v}},\widetilde{\alpha})\in W_m,
\end{aligned} \tag{1.55}$$

where $G_m = (\bar{\mathbf{v}}_m\nabla)\bar{\mathbf{v}}_m$, $G_m^0 = (\bar{\mathbf{v}}_m\nabla)\mathbf{v}_s$, $G_m^1 = (\mathbf{v}_s\nabla)\bar{\mathbf{v}}_m$ and $H_m = f(\bar{\mathbf{v}}_m,\bar{\phi}_{0m})$. We now apply the Fourier transform to the equation (1.55) and take $(\widehat{\mathbf{v}}_m,\widehat{\phi}_{0m})$ as a test function, it yields

$$\begin{aligned}
&2i\pi\tau\int_\Omega |\widehat{\mathbf{v}}_m(\tau)|^2\ d\mathbf{x} + \nu\int_\Omega \nabla\widehat{\mathbf{v}}_m(\tau) : \nabla\widehat{\mathbf{v}}_m(\tau)\ d\mathbf{x} + \int_\Omega \widehat{G}_m(\tau)\cdot\widehat{\mathbf{v}}_m(\tau)\ d\mathbf{x} \\
&\quad + \int_\Omega \widehat{G}_m^0(\tau)\cdot\widehat{\mathbf{v}}_m(\tau)\ d\mathbf{x} + \int_\Omega \widehat{G}_m^1(\tau)\cdot\widehat{\mathbf{v}}_m(\tau)\ d\mathbf{x} \\
&\quad = \int_\Omega \bar{\mathbf{v}}_m(0)\cdot\widehat{\mathbf{v}}_m(\tau)\ d\mathbf{x} - \int_\Omega \bar{\mathbf{v}}_m(T)\cdot\widehat{\mathbf{v}}_m(\tau)e^{-2i\pi\tau T}\ d\mathbf{x} + \widehat{\phi}_{0m}\widehat{H}_m.
\end{aligned} \tag{1.56}$$

where $\widehat{G}_m,\ \widehat{G}_m^0,\ \widehat{G}_m^1$ and $\widehat{H}_m$ are respectively the Fourier transform with respect to time of $G_m,\ G_m^0,\ G_m^1$ and H_m. Note that

$$\begin{aligned}
\widehat{\phi}_{0m}\widehat{H}_m &= a\widehat{\phi}_{0m}\widehat{(\phi_{0m}^2)} + b(\widehat{\phi}_{0m})^2 - \sigma_0\widehat{F}_m - \nu\lambda_1\left((\widehat{\phi}_{0m})^2\|\mathbf{w}\|^2 + 2\widehat{\phi}_{0m}\langle\mathbf{w},\widehat{\mathbf{z}}_m\rangle\right) \\
&= a\widehat{\phi}_{0m}\widehat{(\phi_{0m}^2)} + b(\widehat{\phi}_{0m})^2 - \sigma_0\widehat{F}_m - \ \nu\lambda_1\left(\|\widehat{\mathbf{v}}_m\|^2 - \|\widehat{\mathbf{z}}_m\|^2\right),
\end{aligned} \tag{1.57}$$

where $\widehat{F}_m$ is the Fourier transform with respect to time of $\phi_{0m}\|\mathbf{v}_m\|^2$.

Thanks to lemma 2.2, we have

$$|\widehat{\phi}_{0m}(\tau)| \leq C_b \|\widehat{\mathbf{v}}_m(\tau)\|.$$

By using (1.57) in (1.56) and taking the imaginary part of (1.56) leads to

$$\begin{aligned}|\tau|\|\widehat{\mathbf{v}}_m(\tau)\|^2 &\leq C\,\|\widehat{\mathbf{v}}_m(\tau)\|\left(\sup_{\tau\in\mathbb{R}}\widehat{(\phi_{0m}^2)} + \sup_{\tau\in\mathbb{R}}\widehat{F}_m + \|\bar{\mathbf{v}}_m(T)\| + \|\bar{\mathbf{v}}_m(0)\|\right)\\ &+ C\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{V}(\Omega)}\left(\|\widehat{G}_m(\tau)\|_{\mathbf{V}'(\Omega)} + \|\widehat{G}^0_m(\tau)\|_{\mathbf{V}'(\Omega)} + \|\widehat{G}^1_m(\tau)\|_{\mathbf{V}'(\Omega)}\right). \qquad (1.58)\end{aligned}$$

Note that in the sequel, C stands for different positive constants.

We now prove that the right hand side of (1.58) is bounded.

First, we have

$$\|G_m\|_{\mathbf{V}'(\Omega)} \leq c_1\|\mathbf{v}_m\|^2_{\mathbf{H}^1(\Omega)}, \qquad \|G^s_m\|_{\mathbf{V}'(\Omega)} \leq c_2\|\mathbf{v}_m\|_{\mathbf{H}^1(\Omega)}, \quad s = 0, 1,$$

and thanks to the energy estimate (1.51) satisfied by $\mathbf{v}_m$, G_m and G^s_m remain bounded in $L^1(\mathbb{R}; \mathbf{V}'(\Omega))$ and the functions $\widehat{G}_m$, $\widehat{G}^s_m$ are bounded in $L^\infty(\mathbb{R}; \mathbf{V}'(\Omega))$. Consequently, we have

$$\sup_{\tau\in\mathbb{R}}(\|\widehat{G}_m(\tau)\|_{\mathbf{V}'(\Omega)} + \|\widehat{G}^0_m(\tau)\|_{\mathbf{V}'(\Omega)} + \|\widehat{G}^1_m(\tau)\|_{\mathbf{V}'(\Omega)}) \leq C,$$

and the second line of (1.58) is hence bounded.

We now show that the first four terms in the right hand side of (1.58) are bounded. Thanks to lemma 2.2 and estimate (1.51), ϕ^2_{0m} and $F_m = \phi_{0m}\|\mathbf{v}_m\|^2$ are bounded in $L^1(\mathbb{R})$, and hence $\widehat{\phi^2_{0m}}$ and $\widehat{F}_m$ are bounded in $L^\infty(\mathbb{R})$ with :

$$\sup_{\tau\in\mathbb{R}}\widehat{(\phi^2_{0m})} \leq C \quad \text{and} \quad \sup_{\tau\in\mathbb{R}}\widehat{F}_m \leq C.$$

Thanks to the energy estimate (1.51-a) satisfied by $\mathbf{v}_m$, we have $\|\mathbf{v}_m(T)\| \leq C$ and $\|\mathbf{v}_m(0)\| \leq C$. Inequation (1.58) thus finally reduces to

$$\begin{aligned}|\tau|\|\widehat{\mathbf{v}}_m(\tau)\|^2 &\leq C(\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)} + \|\widehat{\mathbf{v}}_m(\tau)\|)\\ &\leq C\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)},\end{aligned}$$

where C stands for different positive constants.

For $0 < \gamma < \frac{1}{4}$, we now estimate the norm

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \|\widehat{\mathbf{v}}_m(\tau)\|^2 d\tau. \tag{1.59}$$

Note that, (see [20])

$$|\tau|^{2\gamma} \leq c(\gamma) \frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}, \ \forall \tau \in \mathbb{R}.$$

Consequently, we deduce

$$\begin{aligned}
&\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \|\widehat{\mathbf{v}}_m(\tau)\|^2 d\tau \\
&\quad \leq \ c(\gamma) \int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|^2}{1+|\tau|^{1-2\gamma}} d\tau + c(\gamma) \int_{-\infty}^{+\infty} \frac{|\tau| \|\widehat{\mathbf{v}}_m(\tau)\|^2}{1+|\tau|^{1-2\gamma}} d\tau \\
&\quad \leq \ c_3(\gamma) \int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|^2_{\mathbf{H}^1(\Omega)}}{1+|\tau|^{1-2\gamma}} d\tau + c_4(\gamma) \int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}}{1+|\tau|^{1-2\gamma}} d\tau \\
&\quad \leq \ c_3(\gamma) \int_{-\infty}^{+\infty} \|\widehat{\mathbf{v}}_m(\tau)\|^2_{\mathbf{H}^1(\Omega)} d\tau + c_4(\gamma) \int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}}{1+|\tau|^{1-2\gamma}} d\tau.
\end{aligned} \tag{1.60}$$

The last integral in the right hand side of (1.60) satisfies

$$\int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}}{1+|\tau|^{1-2\gamma}} d\tau \leq \left(\int_{-\infty}^{+\infty} \frac{1}{(1+|\tau|^{1-2\gamma})^2} d\tau \right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} \|\widehat{\mathbf{v}}_m(\tau)\|^2_{\mathbf{H}^1(\Omega)} d\tau \right)^{\frac{1}{2}}, \tag{1.61}$$

and the first integral in the right hand side of (1.61) is convergent for any $0 < \gamma < \frac{1}{4}$. On the other hand, using the Parseval equality leads to

$$\int_{-\infty}^{+\infty} \|\widehat{\mathbf{v}}_m(\tau)\|^2_{\mathbf{H}^1(\Omega)} d\tau = \int_0^T \|\mathbf{v}_m(t)\|^2_{\mathbf{H}^1(\Omega)} dt \leq C.$$

Then, the sequence $\mathbf{v}_m$ is bounded in $H^\gamma(0,T;\mathbf{V}(\Omega),\mathbf{H}(\Omega))$, for $0 \leq \gamma \leq \frac{1}{4} - \epsilon$. □

Now, applying Lemmas 3.5 and 3.6, there is a subsequence of $(\mathbf{v}_m)_{m\in\mathbb{N}}$ which converges strongly in $L^2(0,T,\mathbf{H}(\Omega))$.

3.4.3 Passage to the limit

The compactness result obtained in the previous section implies the following strong convergence (at least for a subsequence of $\mathbf{v}_m$ still denoted $\mathbf{v}_m$)

$$\mathbf{v}_m \to \mathbf{v} \text{ strongly in } L^2(0,T;\mathbf{L}^2(\Omega)).$$

This convergence result together with (1.52) enable us to pass to the limit in the following weak formulation, obtained from (1.46) by multiplication by $\varphi \in \mathcal{D}(]0,T[)$ and integration by parts with respect to time

$$
\begin{aligned}
&-\int_0^T \int_\Omega \mathbf{v}_m \cdot \widetilde{\mathbf{v}}_j \varphi'(t)\, d\mathbf{x}dt + \nu \int_0^T \int_\Omega \nabla \mathbf{v}_m : \nabla \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt \\
&\quad + \int_0^T \int_\Omega (\mathbf{v}_m \cdot \nabla \mathbf{v}_m) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt + \int_0^T \int_\Omega (\mathbf{v}_m \cdot \nabla \mathbf{v}_s) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt \\
&\qquad + \int_0^T \int_\Omega (\mathbf{v}_s \cdot \nabla \mathbf{v}_m) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt - \int_\Omega \mathbf{v}_m(0) \widetilde{\mathbf{v}}_j \varphi(0)\, d\mathbf{x} \\
&\qquad\quad = \int_0^T \widetilde{\alpha}_j\, f(\mathbf{v}_m, \phi_{0m}) \varphi(t)\, dt \quad \forall (\widetilde{\mathbf{v}}_j, \widetilde{\alpha}_j) \in W_m.
\end{aligned} \tag{1.62}
$$

Using the weak estimates (1.52) leads to

$$
\begin{aligned}
\int_0^T \int_\Omega \mathbf{v}_m \cdot \widetilde{\mathbf{v}}_j \varphi'(t)\, d\mathbf{x}dt &\xrightarrow[m \to +\infty]{} \int_0^T \int_\Omega \mathbf{v} \cdot \widetilde{\mathbf{v}}_j \varphi'(t)\, d\mathbf{x}dt, \\
\int_0^T \int_\Omega \nabla \mathbf{v}_m : \nabla \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt &\xrightarrow[m \to +\infty]{} \int_0^T \int_\Omega \nabla \mathbf{v} : \nabla \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt, \\
\int_0^T \int_\Omega (\mathbf{v}_m \cdot \nabla \mathbf{v}_s) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt &\xrightarrow[m \to +\infty]{} \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}_s) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt, \\
\int_0^T \int_\Omega (\mathbf{v}_s \cdot \nabla \mathbf{v}_m) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt &\xrightarrow[m \to +\infty]{} \int_0^T \int_\Omega (\mathbf{v}_s \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt,
\end{aligned}
$$

for the linear terms. Further, since $\mathbf{v}_m$ converges to $\mathbf{v}$ in $L^2(0,T;\mathbf{V}(\Omega))$ weakly, and in $L^2(0,T;\mathbf{L}^2(\Omega))$ strongly, we can pass to the limit in the nonlinear term to obtain

$$
\int_0^T \int_\Omega (\mathbf{v}_m \cdot \nabla \mathbf{v}_m) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt \xrightarrow[m \to +\infty]{} \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt. \tag{1.63}
$$

Using Lemma 2.2 and according to (1.51-a), $\phi_{0m} \in L^\infty(0,T)$. Then for a subsequence of ϕ_{0m} (still denoted by ϕ_{0m}) :

$$
\phi_{0m} \rightharpoonup \alpha \text{ weakly}^* \text{ in } L^\infty(0,T). \tag{1.64}
$$

As far as the right hand side of (1.62) is concerned. Let us notice that the convergence of $\mathbf{v}_m$ in $L^2([0,T] \times \Omega)$ implies its convergence in $L^1(0,T;\mathbf{L}^2(\Omega))$. Hence

$$
\|\mathbf{v}_m\| \quad \longrightarrow \quad \|\mathbf{v}\| \text{ in } L^1(0,T). \tag{1.65}
$$

Due to lemma 2.2, we have

$$|\phi_{0p} - \phi_{0q}| \leq C_b \|\mathbf{v}_p - \mathbf{v}_q\|, \quad \forall (\mathbf{v}_p, \phi_{0p}), (\mathbf{v}_q, \phi_{0q}) \in W_m,$$

and ϕ_{0m} is then a Cauchy sequence in $L^1(0,T)$ and

$$\phi_{0m} \quad \longrightarrow \quad \phi_0 \text{ in } L^1(0,T). \tag{1.66}$$

Further, according to (1.64) we have $\phi_0 = \alpha \in L^\infty(0,T)$ from [12, Proposition II.1.26]. Since $\|\mathbf{v}_m\|$ and ϕ_{0m} are bounded in $L^\infty(0,T)$, using (1.65) and (1.66) we obtain

$$\begin{array}{cccl} \|\mathbf{v}_m\| & \longrightarrow & \|\mathbf{v}\| & \text{in } L^p(0,T), \\ \phi_{0m} & \longrightarrow & \alpha & \text{in } L^p(0,T), \end{array}$$

from [12, Corollaire II.1.24], for all $p \in]1, +\infty[$.

Now we can pass to the limit in the following terms :

$$\int_0^T \tilde{\alpha}_j \phi_{0m}^2 \varphi(t) \quad \xrightarrow[m\to+\infty]{} \quad \int_0^T \tilde{\alpha}_j \alpha^2 \varphi(t), \tag{1.67}$$

$$\int_0^T \tilde{\alpha}_j \phi_{0m} \|\mathbf{v}_m\|^2 \varphi(t) \quad \xrightarrow[m\to+\infty]{} \quad \int_0^T \tilde{\alpha}_j \alpha \|\mathbf{v}\|^2 \varphi(t), \tag{1.68}$$

$$\int_0^T \tilde{\alpha}_j \langle \mathbf{w}, \mathbf{z}_m \rangle \varphi(t) \quad \xrightarrow[m\to+\infty]{} \quad \int_0^T \tilde{\alpha}_j \langle \mathbf{w}, \mathbf{z} \rangle \varphi(t), \tag{1.69}$$

because $\mathbf{z}_m = \mathbf{v}_m - \phi_{0m}\mathbf{w}$. Consequently

$$\int_0^T \widetilde{\alpha}_j f(\mathbf{v}_m, \phi_{0m}) \varphi(t) dt \quad \xrightarrow[m\to+\infty]{} \quad \int_0^T \widetilde{\alpha}_j f(\mathbf{v}, \alpha) \varphi(t) dt,$$

where

$$f(\mathbf{v}, \alpha) = a\alpha^2 + b\alpha - \sigma_0 \|\mathbf{v}\|^2 \alpha - \nu\lambda_1 \|\mathbf{w}\|^2 \alpha - 2\nu\lambda_1 \langle \mathbf{w}, \mathbf{z} \rangle.$$

Passing to the limit in (1.62) then gives

$$\begin{aligned} -\int_0^T \int_\Omega \mathbf{v} \cdot \widetilde{\mathbf{v}} \varphi'(t)\, d\mathbf{x}dt + \nu \int_0^T \int_\Omega \nabla \mathbf{v} : \nabla \widetilde{\mathbf{v}} \varphi(t)\, d\mathbf{x}dt + \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}} \varphi(t)\, d\mathbf{x}dt \\ + \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}_s) \cdot \widetilde{\mathbf{v}} \varphi(t)\, d\mathbf{x}dt + \int_0^T \int_\Omega (\mathbf{v}_s \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}} \varphi(t)\, d\mathbf{x}dt - \int_\Omega \mathbf{v}_0 \widetilde{\mathbf{v}} \varphi(0)\, d\mathbf{x} \\ = \int_0^T \widetilde{\alpha} f(\mathbf{v}, \alpha) \varphi(t)\, dt. \end{aligned} \tag{1.70}$$

for all $\widetilde{\mathbf{v}} = \widetilde{\mathbf{v}}_j,\ \forall j = 0, 1, 2, \cdots, m$. By linearity, equation (1.70) holds true for all $\widetilde{\mathbf{v}}$ combination of finite $\widetilde{\mathbf{v}}_j$ and by density, for any element of $W(Q)$. $\square$

Finally, it remains to retrieve the stabilized problem (1.24), which requires to prove the existence of pressure.

3.5 Existence of the Pressure

First, we recall a result obtained in [25]

Lemma 3.7. *Let* $\mathbf{f} \in \mathcal{D}'(]0,T[;\mathbf{H}^{-1}(\Omega))$ *such that* $\langle \mathbf{f}, \widetilde{\mathbf{v}} \rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} = 0 \ \forall \widetilde{\mathbf{v}} \in \mathbf{V}_0(\Omega)$. *Then there exists* $q \in \mathcal{D}'(]0,T[;\mathbf{L}^2(\Omega))$ *such that* $\mathbf{f} = \nabla q$.

This lemma is utilized to prove the following.

Lemma 3.8. *There exists* $p \in \mathcal{D}'(]0,T[;L^2(\Omega))$ *such that* $(\mathbf{v}, p)$ *satisfies (1.24-a) in the distribution sense.*

Proof. By choosing $\varphi \in \mathcal{D}(0,T)$ in (1.70), we obtain

$$\int_0^T \int_\Omega \frac{\partial \mathbf{v}}{\partial t} \cdot \widetilde{\mathbf{v}} \varphi(t) \, d\mathbf{x}dt + \nu \int_0^T \int_\Omega \nabla \mathbf{v} : \nabla \widetilde{\mathbf{v}} \varphi(t) \, d\mathbf{x}dt + \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}} \varphi(t) \, d\mathbf{x}dt$$
$$+ \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}_s) \cdot \widetilde{\mathbf{v}} \varphi(t) \, d\mathbf{x}dt + \int_0^T \int_\Omega (\mathbf{v}_s \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}} \varphi(t) \, d\mathbf{x}dt$$
$$= \int_0^T \widetilde{\alpha} f(\mathbf{v}, \alpha) \varphi(t) dt, \qquad \forall (\widetilde{\mathbf{v}}, \widetilde{\alpha}) \in W(Q). \tag{1.71}$$

Further, taking $\widetilde{\alpha} = 0$ leads to

$$\int_\Omega \frac{\partial \mathbf{v}}{\partial t} \cdot \widetilde{\mathbf{v}} \, d\mathbf{x} + \nu \int_\Omega \nabla \mathbf{v} : \nabla \widetilde{\mathbf{v}} \, d\mathbf{x} + \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}} \, d\mathbf{x}$$
$$+ \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}_s) \cdot \widetilde{\mathbf{v}} \, d\mathbf{x} + \int_\Omega (\mathbf{v}_s \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}} \, d\mathbf{x} = 0, \text{ in } \mathcal{D}'(0,T). \tag{1.72}$$

Then, letting

$$\mathbf{f} = \frac{\partial \mathbf{v}}{\partial t} - \nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}_s + (\mathbf{v}_s \cdot \nabla) \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v},$$

and using (1.72), we obtain $\mathbf{f} \in \mathcal{D}'(]0,T[;\, \mathbf{H}^{-1}(\Omega))$ and $\langle \mathbf{f}, \widetilde{\mathbf{v}} \rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} = 0, \ \forall \widetilde{\mathbf{v}} \in \mathbf{V}_0(\Omega)$. Finally, using Lemma 3.7, there exists $p \in \mathcal{D}'(]0,T[;\, L^2(\Omega))$ such that $\mathbf{f} = -\nabla p$. $\square$

Now, we prove that $(\mathbf{v}, p)$ satisfies (1.24-f). Let us first define the space

$$E(\Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \text{div } \mathbf{u} \in L^2(\Omega)\},$$

and recall the following Lemma obtained in [25, Chap I, Theorem 1.2] :

Lemma 3.9. *Let Ω be an open bounded set of class $\mathcal{C}^2$. Then there exists a linear continuous operator $\gamma_n \in \mathcal{L}(E(\Omega), H^{-1/2}(\Gamma))$ such that*

$$\gamma_n \mathbf{u} = \textit{ the restriction of } \mathbf{u}\cdot\mathbf{n} \textit{ to } \Gamma, \textit{ for every } \mathbf{u} \in \mathcal{D}(\bar{\Omega}).$$

The following generalized Stokes formula is true for all $\mathbf{u} \in E(\Omega)$ and $\mathbf{w} \in \mathbf{H}^1(\Omega)$,

$$(\mathbf{u}, \nabla \mathbf{w}) + (div\ \mathbf{u}, \mathbf{w}) = \langle \gamma_n \mathbf{u}, \gamma_0 \mathbf{w}\rangle, \tag{1.73}$$

where $\gamma_0 \in \mathcal{L}(\mathbf{H}^1(\Omega), \mathbf{L}^2(\Gamma))$ is the trace operator.

By writing (1.24-a) in the form

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{div}(-\nu\nabla\mathbf{v} + Ip) + (\mathbf{v}\cdot\nabla)\mathbf{v}_s + (\mathbf{v}_s\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = 0 \ \text{ in } Q,$$

and using Lemma 3.9, we obtain

$$\begin{aligned}\int_\Omega \frac{\partial \mathbf{v}}{\partial t}\cdot\widetilde{\mathbf{v}}\ d\mathbf{x} + \int_\Omega (\nu\nabla\mathbf{v} - Ip) : \nabla\widetilde{\mathbf{v}}\ d\mathbf{x} + \langle(-\nu\nabla\mathbf{v} + Ip)\cdot\mathbf{n}, \widetilde{\mathbf{v}}\rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} \\ + \int_\Omega (\mathbf{v}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\ d\mathbf{x} + \int_\Omega (\mathbf{v}\cdot\nabla\mathbf{v}_s)\cdot\widetilde{\mathbf{v}}\ d\mathbf{x} + \int_\Omega (\mathbf{v}_s\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\ d\mathbf{x} = 0,\end{aligned}$$

$\forall(\widetilde{\mathbf{v}}, \widetilde{\alpha}) \in W(Q)$. Since $(\widetilde{\mathbf{v}}, \widetilde{\alpha}) \in W(Q)$, we have

$$\begin{aligned} pI : \nabla\widetilde{\mathbf{v}} = p\nabla\cdot\widetilde{\mathbf{v}} = 0, \\ \langle(-\nu\nabla\mathbf{v} + Ip)\cdot\mathbf{n}, \widetilde{\mathbf{v}}\rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} = -\widetilde{\alpha}\int_{\Gamma_b} [\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}]\cdot\mathbf{g}\ d\zeta.\end{aligned}$$

Consequently,

$$\begin{aligned}\int_\Omega \frac{\partial \mathbf{v}}{\partial t}\cdot\widetilde{\mathbf{v}}\ d\mathbf{x} + \nu\int_\Omega \nabla\mathbf{v} : \nabla\widetilde{\mathbf{v}}\ d\mathbf{x} + \int_\Omega (\mathbf{v}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\ d\mathbf{x} + \int_\Omega (\mathbf{v}\cdot\nabla\mathbf{v}_s)\cdot\widetilde{\mathbf{v}}\ d\mathbf{x} \\ + \int_\Omega (\mathbf{v}_s\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\ d\mathbf{x} = \widetilde{\alpha}\int_{\Gamma_b} [\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}]\cdot\mathbf{g}\ d\zeta.\end{aligned} \tag{1.74}$$

By comparing (1.71) and (1.74), we deduce

$$\int_{\Gamma_b} [\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}]\cdot\mathbf{g}\ d\zeta = f(\mathbf{v}, \alpha).$$

Finally, it remains to verify the initial condition. In this purpose, firstly, we let

$$\mathcal{E}(Q) = \big\{(\mathbf{v}, \alpha) \in \mathbf{H}(\Omega) \times \mathbb{R}, \text{ such that } \mathbf{v} \cdot \mathbf{n} = \alpha \mathbf{g} \cdot \mathbf{n} \text{ on } \Gamma_b\big\}, \tag{1.75}$$

and we obtain the following Lemma

Lemma 3.1. *The space $W(Q)$ is dense in $\mathcal{E}(Q)$.*

Proof. We define

$$\mathcal{G}(Q) \quad = \quad \big\{(\mathbf{u}, \alpha) \in \mathbf{H}^1(\Omega) \times \mathbb{R} \;: \mathbf{u} = 0 \text{ on } \Gamma_l, \; \mathbf{u} = \alpha \mathbf{g} \text{ on } \Gamma_b\big\}. \tag{1.76}$$

By construction, we have

$$\mathbf{H}_0^1(\Omega) \times \mathbb{R} \subset \mathcal{G}(Q) \subset \mathbf{L}^2(\Omega) \times \mathbb{R}, \tag{1.77}$$

$$W(Q) = \mathcal{G}(Q) \cap \mathcal{E}(Q), \tag{1.78}$$

$$\mathcal{E}(Q) = \mathcal{E}(Q) \cap \mathbf{L}^2(\Omega) \times \mathbb{R}. \tag{1.79}$$

Since $\mathbf{H}_0^1(\Omega)$ is dense in $\mathbf{L}^2(\Omega)$, according to (1.77) we have $\mathcal{G}(Q)$ dense in $\mathbf{L}^2(\Omega) \times \mathbb{R}$ and hence, thanks to (1.78)-(1.79), the space $W(Q)$ is dense in $\mathcal{E}(Q)$. $\square$

Secondly, we multiply (1.24-a) by $\widetilde{\mathbf{v}}\varphi$ with $\varphi(T) = 0$ and integrate with respect to time and space

$$\begin{aligned} -\int_0^T \int_\Omega \mathbf{v} \cdot \widetilde{\mathbf{v}} \varphi'(t) \, d\mathbf{x}dt + \nu \int_0^T \int_\Omega \nabla \mathbf{v} : \nabla \widetilde{\mathbf{v}} \varphi(t) \, d\mathbf{x}dt \\ + \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}} \varphi(t) \, d\mathbf{x}dt + \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}_s) \cdot \widetilde{\mathbf{v}} \varphi(t) \, d\mathbf{x}dt \\ + \int_0^T \int_\Omega (\mathbf{v}_s \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}} \varphi(t) \, d\mathbf{x}dt - \int_\Omega \mathbf{v}(0) \widetilde{\mathbf{v}} \varphi(0) \, d\mathbf{x} \\ = \int_0^T \widetilde{\alpha} f(\mathbf{v}, \alpha) \varphi(t) \, dt. \end{aligned} \tag{1.80}$$

By comparing (1.70) and (1.80), we obtain $\int_\Omega (\mathbf{v}(0) - \mathbf{v}_0) \cdot \widetilde{\mathbf{v}} \varphi(0) \, d\mathbf{x} = 0$, and choosing φ such that $\varphi(0) = 1$, leads to

$$\int_\Omega (\mathbf{v}(0) - \mathbf{v}_0) \cdot \widetilde{\mathbf{v}} \, d\mathbf{x} = 0, \quad \forall (\widetilde{\mathbf{v}}, \widetilde{\alpha}) \in W(Q). \tag{1.81}$$

From (1.81) and Lemma 3.1 we obtain $\mathbf{v}(0) = \mathbf{v}_0$ in $\mathcal{E}(Q)$.

Proposition 3.1. *When* $\mathbf{v}$ *is solution of the stabilization problem (1.24), for a given initial perturbation* $\mathbf{v}_0 \in \mathbf{H}(\Omega)$ *and profile* $\mathbf{g} \in \mathbf{V}^{1/2}(\Gamma_b)$ *such that* $\alpha_0 \mathbf{g} \cdot \mathbf{n} = \mathbf{v}_0 \cdot \mathbf{n} \neq 0$ *on* Γ_b*, the control is not identically zero. i.e.* $\alpha \not\equiv 0$*.*

The proof of Proposition 3.1 is given after Lemmas 3.2 is established. We start by giving the following functionals spaces

$$\mathcal{H}(\Omega) \;=\; \left\{\mathbf{u} \in \mathbf{L}^2(\Omega) \;:\nabla \cdot \mathbf{u} = 0,\;\; \mathbf{u} \cdot \mathbf{n} = 0 \;\text{ on } \Gamma\right\}, \tag{1.82}$$

$$\mathbf{V}^s(\Omega) \;=\; \text{the closure } \boldsymbol{\mathcal{V}}(\Omega) \text{ in } \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^s(\Omega),\; s \geq \frac{d}{2}. \tag{1.83}$$

Lemma 3.2. *Let* $\mathbf{v}$ *satisfies the stabilization problem (1.24) with* $\alpha \equiv 0$*,*

1. *In 2-dimensional space,*

$$\mathbf{v} \in C^0([0,T], \mathcal{H}(\Omega)). \tag{1.84}$$

2. *In 3-dimensional space,*

$$\mathbf{v} \in C^0([0,T], \mathcal{H}_{weak}(\Omega)), \tag{1.85}$$

namely, $\mathbf{v}$ *is weakly continuous from* $[0,T]$ *into* $\mathcal{H}(\Omega)$*.*

Proof. According to the variation formulation (1.26), the solution $\mathbf{v}$ satisfies

$$-\langle d_t \mathbf{v}, \mathbf{z}\rangle = \nu a(\mathbf{v}, \mathbf{z}) + b(\mathbf{v}, \mathbf{v}_s, \mathbf{z}) + b(\mathbf{v}_s, \mathbf{v}, \mathbf{z}) + b(\mathbf{v}, \mathbf{v}, \mathbf{z}), \quad \forall \mathbf{z} \in \mathbf{V}_0(\Omega) \tag{1.86}$$

and

$$\langle d_t \mathbf{v}, \mathbf{w}\rangle + \nu a(\mathbf{v}, \mathbf{w}) + b(\mathbf{v}, \mathbf{v}_s, \mathbf{w}) + b(\mathbf{v}_s, \mathbf{v}, \mathbf{w}) + b(\mathbf{v}, \mathbf{v}, \mathbf{w}) = f(\mathbf{v}, \alpha). \tag{1.87}$$

If $\alpha \equiv 0$, according to estimates (1.30)-(1.31), we firstly obtain

$$\mathbf{v} \in L^\infty(]0,T[, \mathcal{H}(\Omega)) \cap L^2(]0,T[, \mathbf{V}_0(\Omega)). \tag{1.88}$$

Secondly, by taking the supremum of (1.86) with respect to $\mathbf{z} \in \mathbf{V}_0(\Omega)$ with $\|\mathbf{z}\|_{\mathbf{V}_0(\Omega)} = 1$, and applying inequality (4.4) in [25, Lemma 4.1 Page 217], leads to

$$\begin{cases} \dfrac{d\mathbf{v}}{dt} \in L^2(]0,T[, \mathbf{V}^{-1}(\Omega)) \;\text{ if }\; d = 2, \\ \dfrac{d\mathbf{v}}{dt} \in L^2(]0,T[, \mathbf{V}^{-\frac{3}{2}}(\Omega)) \;\text{ if }\; d = 3 \end{cases} \tag{1.89}$$

where $\mathbf{V}^{-\frac{3}{2}}(\Omega) = (\mathbf{V}^{\frac{3}{2}}(\Omega))'$. Finally, as in [12, Proposition IV.1.7 Page 217], due to (1.88)-

(1.89) we obtain (1.84)-(1.85). □

Proof of the Proposition 3.1. Assume, by absurd, that $\alpha \equiv 0$. Recall that from (1.81) and Lemma 3.1 we have

$$\mathbf{v}(0) = \mathbf{v}_0 \ \text{ in } \ \mathcal{E}(Q). \tag{1.90}$$

1. In 2-dimensional space, from (1.84), $\mathbf{v} \in C^0([0,T], \mathcal{H}(\Omega))$. i.e.

$$\mathbf{v}(t) \longrightarrow \mathbf{v}(0) = \mathbf{v}_0 \ \text{ in } \ \mathcal{H}(\Omega), \ \text{ when } \ t \to 0^+,$$

which is impossible since $\mathbf{v}_0 \cdot \mathbf{n} \neq 0$ on Γ_b.

2. In 3-dimensional space, according to (1.85), $\mathbf{v} \in C^0([0,T], \mathcal{H}_{weak}(\Omega))$. Firstly, by the weak continuity, we have

$$\|\mathbf{v}_0\|^2 \leq \text{lim inf}_{t\to 0^+} \|\mathbf{v}(t)\|^2.$$

Secondly, by the energy inequality (1.30),

$$\text{lim sup}_{t\to 0^+} \|\mathbf{v}(t)\|^2 \leq \|\mathbf{v}_0\|^2.$$

Hence $\lim_{t\to 0^+} \|\mathbf{v}(t)\| = \|\mathbf{v}_0\|$, sufficient condition (thanks to the weak continuity) to prove the strong continuity of v on 0 i.e.

$$\mathbf{v}(t) \longrightarrow \mathbf{v}(0) = \mathbf{v}_0 \ \text{ in } \ \mathcal{H}(\Omega), \ \text{ when } \ t \to 0^+,$$

which is impossible since $\mathbf{v}_0 \cdot \mathbf{n} \neq 0$ on Γ_b.

□

4 Concluding remarks

In this work the exponential stabilization of the two and three-dimensional Navier-Stokes equations in a bounded domain is studied around a given steady-state flow, using a boundary feedback control. In order to determine a feedback law, an extended system coupling the Navier-Stokes equations with an equation satisfied by the control on the domain boundary is considered. We first assume that on Σ_b (a part of the domain boundary), the trace of the fluid velocity is proportional to a given velocity profile g. The proportionality coefficient α measures the velocity flux at the interface, it is an unknown of the problem and is written in feedback form. By using the Galerkin method, α is determined such that the Dirichlet boundary control $\mathbf{u}_b = \alpha \mathbf{g}$ is satisfied on Σ_b, and the stabilizing boundary control is built. The resulting nonlinear feedback control is proven to be globally exponentially stabilizing the steady states of the two and three-dimensional Navier-Stokes equations. This feedback control was shown to guarantee global stability in the L^2-norm.

Finally, in order to take into account (1.24-f) in the variational formulation, the test functions, for example $\widetilde{\mathrm{v}}$, need to be written on the form $\widetilde{\mathrm{v}} = \widetilde{\alpha}\mathrm{g}$. This requires to construct a finite-element basis which allows such a requirement and hence at least one element of the basis, for example w, such that $\mathrm{w} = \mathrm{g}$ on Γ_b. A number of choices, including both continuous and discontinuous approximations, may be investigated. Once the finite-element basis is obtained, equation (1.24-f), satisfied by the control, will be present in the discrete ODE. A priori, the control α should be robust, since it is bounded by the perturbation (see inequality (15) in Lemma 2.2), and numerically efficient. In a forthcoming paper, several test cases are performed and discussed.

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Chapitre 2

Boundary stabilization of the Navier-Stokes Model with feedback controller around a non-stationary state

Abstract

This paper presents a boundary feedback control for the two and three-dimensional Navier-Stokes equations in a bounded domain Ω around a given *non-stationary velocity*. In order to determine a feedback control law, we consider an extended system coupling the equations governing the perturbation with an equation satisfied by the control on the domain boundary. By using the Faedo-Galerkin method and a priori estimation techniques, a stabilizing boundary control is built. This control law ensures a decrease of the energy of the controlled discrete system. A compactness result then allows us to pass to the limit in the system satisfied by the approximated solutions.

Keywords : Navier-Stokes system, feedback control, boundary stabilization, Galerkin method.

1 Introduction

This paper presents a boundary feedback control for the two and three-dimensional Navier-Stokes equations in a bounded domain Ω around a given non-stationary velocity. Let Ω be a bounded and connected domain in $\mathbb{R}^d$ $(d = 2, 3)$, with a boundary Γ of class C^2, and composed of two connected components Γ_l and Γ_b such that $\Gamma = \Gamma_l \cup \Gamma_b$. In particular, the boundary Γ_b is the part of Γ, where a Dirichlet boundary control in feedback form has to be determined. Let $T > 0$ a fixed real number, we take $Q = [0, T[\times\Omega$, $\Sigma_l = [0, T[\times\Gamma_l$, $\Sigma_b = [0, T[\times\Gamma_b$ and we consider the trajectory $(\boldsymbol{\psi}, q)$ solution of the non-stationary Navier-

Stokes equations

$$\begin{cases} \dfrac{\partial \boldsymbol{\psi}}{\partial t} - \nu\Delta\boldsymbol{\psi} + (\boldsymbol{\psi}\cdot\nabla)\boldsymbol{\psi} + \nabla q = \mathbf{f} & \text{in } Q, \\ \nabla\cdot\boldsymbol{\psi} = 0 & \text{in } Q, \\ \boldsymbol{\psi} = 0 & \text{on } \Sigma_l, \\ \boldsymbol{\psi} = \boldsymbol{\psi}_b & \text{on } \Sigma_b, \end{cases} \tag{2.1}$$

where $\nu > 0$ is the viscosity, f represents body forces acting on the fluid and ψ_b the boundary condition in Γ_b. Let us first define the set of *admissible target velocities* $\mathcal{U}_{ad}$. The solution $\psi(t,\mathbf{x})$ of (2.1) is said to be in the set admissible target velocities $\mathcal{U}_{ad}$ if

$$\sup_{t\le T} \|\nabla\boldsymbol{\psi}(t,\mathbf{x})\| < \frac{\nu}{C_\Omega}, \tag{2.2}$$

where $\|\cdot\| = \|\cdot\|_{(L^2(\Omega))^d}$, $\mathbf{x} = (x,y,z)$ if $d=3$ and C_Ω is a positive constant defined later in (2.18).

We now consider the perturbed trajectory $(\mathbf{u},r)$ solution of the non-stationary Navier-Stokes equations

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla r = \mathbf{f} & \text{in } Q, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } Q, \\ \mathbf{u} = \mathbf{v}_b + \boldsymbol{\psi}_b & \text{on } \Sigma_b, \\ \mathbf{u} = 0 & \text{on } \Sigma_l, \\ \mathbf{u}(t=0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) + \boldsymbol{\psi}(t=0,\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{2.3}$$

where $\mathbf{v}_b$ is the control input and function $\mathbf{v}_0$ can be viewed as a perturbation of the initial state (2.1). By substituting $(\mathbf{u},r) = (\mathbf{v}+\boldsymbol{\psi}, p+q)$ in (2.3), we obtain the following system

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\boldsymbol{\psi} + (\boldsymbol{\psi}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0 & \text{in } Q, \\ \nabla\cdot\ \mathbf{v} = 0 & \text{in } Q, \\ \mathbf{v} = \mathbf{v}_b & \text{on } \Sigma_b, \\ \mathbf{v} = 0 & \text{on } \Sigma_l, \\ \mathbf{v}(t=0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega. \end{cases} \tag{2.4}$$

The usual function spaces $L^2(\Omega)$, $H^1(\Omega)$, $H^1_0(\Omega)$ are used and we let $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{H}^1(\Omega) = (H^1(\Omega))^d$, $\mathbf{H}^1_0(\Omega) = (H^1_0(\Omega))^d$. Negative ordered Sobolev spaces $\mathbf{H}^{-1}(\Omega)$ is defined as the dual space, i.e. $\mathbf{H}^{-1}(\Omega) = \{\mathbf{H}^1_0(\Omega)\}'$. We denote by $\langle\cdot\mid\cdot\rangle$ and $\|\cdot\| = \|\cdot\|_{\mathbf{L}^2(\Omega)}$, the scalar product and norm in $\mathbf{L}^2(\Omega)$,respectively. Further, if $\mathbf{u} \in \mathbf{L}^2(\Omega)$ is such that $\nabla \cdot \mathbf{u} \in L^2(\Omega)$, we denote the normal trace of $\mathbf{u}$ in $\mathbf{H}^{-\frac{1}{2}}(\Gamma)$ by $\mathbf{u} \cdot \mathbf{n}$, where $\mathbf{n}$ denotes the unit outer normal vector to Γ.

Our goal is the following : for a prescribed rate of decrease $\sigma > 0$, we need to find a feedback control $\mathbf{v}_b$ on Σ_b such that the velocity $\mathbf{v}$ in (2.4) satisfies the exponential decay

$$\|\mathbf{v}(t)\| \leq \|\mathbf{v}_0\| e^{-\sigma(t)} \in (0, \infty). \tag{2.5}$$

Note that $\sigma(t)$ is usually written as $\sigma_0 t$ in previous studies [1, 5, 17, 29], where σ_0 is positive constant.

The control $\mathbf{v}_b(t)$ is called a feedback if there exists a mapping $\mathcal{M} \ : \ \mathbf{X}(\Omega) \to \mathbf{U}(\Gamma_b)$ such that

$$\mathbf{v}_b(t) = \mathcal{M}(\mathbf{v}(t)), \quad t \in (0, \infty), \tag{2.6}$$

where the spaces $\mathbf{X}(\Omega)$ and $\mathbf{U}(\Gamma_b)$ are defined in the sequel.

The theoretical setting of the boundary feedback stabilization procedure, for the non-stationary incompressible Navier-Stokes equations around a given *stationary velocity*, has been studied in a number of papers, e.g. A.V. Fursikov [17, 18], V. Barbu et al. [5, 10, 11, 12, 13], J.-P. Raymond et al. [29, 30, 31] and M. Badra et al. [1, 2, 3]. In these publications, a linear feedback law is first determined by solving a linear control problem, and this linear feedback is then used in order to stabilize the original non linear system. Such a procedure leads to use the Oseen-operator and the target velocity ψ in (2.4) is chosen to be independent of time, i.e. $\psi(t, \mathbf{x}) \equiv \psi(\mathbf{x})$. However, Another approach for stabilizing fluid dynamics equations is proposed in [16, 22, 23, 27, 32]. The method was first published with application on a 1D shallow water equation in [32]. It consists on establishing an equation involving the derivative of energy with respect to time, and the boundary conditions. Then, by utilizing adequate feedback boundary conditions, the authors manage to get the energy's exponential decrease. So far, the method has been applied to stabilize irrigation channel networks [22, 23], coupled shallow-water and erosion-sedimentation equations [16], and the Navier-Stokes system around a steady-state [27]. Note that in [27], an extended system is considered with an additional equation satisfied by the control on the domain boundary, and the boundary feedback control is constructed via a Galerkin method. Thereby, the authors stabilize the Navier-Stokes equations in a bounded domain Ω around a given steady-state which satisfies the stationary Navier-Stokes equations.

In this paper, the approach of [27], using an extended system is followed in order to stabilize the two and three-dimensional Navier-Stokes problem around a given *non-stationary state* $\psi(t, \mathbf{x})$ instead of a stationary state $\psi(\mathbf{x})$ employed in [27]. The boundary control $\mathbf{v}_b$ in (2.4) is rewritten on the form $\mathbf{v}_b = \alpha(t)\mathbf{g}(\mathbf{x})$ on Σ_b, where $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$ is assumed to verify $\mathbf{g} = 0$ on Γ_l, $\mathbf{g}\cdot\mathbf{n} \neq 0$ on Γ_b and $\int_{\Gamma_b} \mathbf{g}\cdot\mathbf{n} = 0$. The proportionality coefficient α is a priori unknown. In order to stabilize (2.4), with $\mathbf{v}_b = \alpha(t)\mathbf{g}(\mathbf{x})$ on Σ_b, by employing energy a priori estimations, the quantity α is found to satisfy the relation

$$\int_{\Gamma_b} [\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}]\cdot\mathbf{g}\, d\zeta = f(\mathbf{v},\alpha), \tag{2.7}$$

where f is a polynomial in α of degree 2, defined later in (2.34). The quantity α depends nonlinearly on $\mathbf{v}$ in (2.7), and hence α satisfies a nonlinear feedback law. Such an exponential boundary feedback stabilization for tracking the *non-stationary velocity* (with $\boldsymbol{\psi}(t,\mathbf{x})$) in the Navier-Stokes equations flows is new, to our knowledge, although the problem has been considered previously in [8] for the internal exponential stabilization case and in [24] with a two-dimensional boundary control only, and for an optimal control problem, i.e. not for an exponential stabilization control.

System (2.4) is then extended by adding (2.7), and the extended system, namely (2.4) and (2.7), with $\mathbf{v}_b = \alpha(t)\mathbf{g}(\mathbf{x})$ on Σ_b, is the stabilization problem considered in this paper, i.e.

$$\begin{cases} (a) & \dfrac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\boldsymbol{\psi} + (\boldsymbol{\psi}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0 & \text{in } Q, \\ (b) & \nabla\cdot\ \mathbf{v} = 0 & \text{in } Q, \\ (c) & \mathbf{v} = \alpha(t)\mathbf{g}(\mathbf{x}), & \text{on } \Sigma_b, \\ (d) & \mathbf{v} = 0 & \text{on } \Sigma_l, \\ (e) & \mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega, \\ (f) & \displaystyle\int_{\Gamma_b} [\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}]\cdot\mathbf{g}\, d\zeta = f(\mathbf{v},\alpha). & \end{cases} \tag{2.8}$$

In order to determined α, leading to the determination of the boundary control $\mathbf{v}_b$, system (2.8) is solved via a Galerkin procedure, as in [27], which consists of building a sequence of approximated solutions using an adequate Galerkin basis.

The paper is organized as follows. In section 2, the notations and mathematical preliminaries are given. In section 3, we build the control law and in section 4, thanks to technics developed in [25] (which are not related specifically to a stabilization problem), the existence of at least one weak solution of the non-linear Navier-Stokes system is established by applying the Galerkin method.

2 Notation and Preliminaries

2.1 Function Spaces

Several spaces of free divergence functions are now introduced :

$$\mathbf{V}(\Omega) = \Big\{\mathbf{u} \in \mathbf{H}^1(\Omega) \ : \ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{u} = 0 \text{ on } \Gamma_l, \int_{\Gamma_b} \mathbf{u}\cdot\mathbf{n}\, d\zeta = 0\Big\}, \tag{2.9}$$

$$\mathbf{V}_0(\Omega) = \{\mathbf{u} \in \mathbf{H}^1_0(\Omega) \ : \ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega\}, \tag{2.10}$$

$$\mathbf{H}(\Omega) = \Big\{\mathbf{u} \in \mathbf{L}^2(\Omega) \ : \nabla\cdot\mathbf{u} = 0,\ \ \mathbf{u}\cdot\mathbf{n} = 0 \ \text{ on } \Gamma_l, \int_{\Gamma_b} \mathbf{u}\cdot\mathbf{n}\, d\zeta = 0\Big\}. \tag{2.11}$$

Since $\mathbf{V}(\Omega)$ is a closed subspace of $\mathbf{H}^1(\Omega)$, we have, by definition $\|\cdot\|_{\mathbf{V}(\Omega)} = \|\cdot\|_{\mathbf{H}^1(\Omega)}$.

Next, we define the pressure space with zero mean value :

$$L^2_0(\Omega) = \Big\{p \in L^2(\Omega),\ \int_\Omega p(\mathbf{x})\, d\mathbf{x} = 0\Big\}.$$

Definition 2.1. *Let $V^{1/2}(\Gamma_b)$ be the space of trace functions that, if extended by zero over Γ, belongs to $\mathbf{H}^{1/2}(\Gamma)$. Further, for $\mathbf{g} \in V^{1/2}(\Gamma_b)$ such that $\mathbf{g}\cdot\mathbf{n} \neq 0$ on Γ_b, we define*

$$W(Q) = \{(\mathbf{v},\alpha) \in \mathbf{V}(\Omega)\times\mathbb{R},\ \textit{such that } \mathbf{v} = \alpha\mathbf{g} \textit{ on } \Gamma_b\}. \tag{2.12}$$

The following Lemma [25], will be used in the sequel.

Lemma 2.2. *There exists a constant $C_b > 0$ such that, for all $(\mathbf{v},\alpha) \in W(Q)$, we have*

$$|\alpha| \le C_b\|\mathbf{v}\|. \tag{2.13}$$

2.2 Linear Forms

In order to define a weak form of the Navier-Stokes equations, we introduce the continuous bilinear form

$$a(\mathbf{v}_1,\mathbf{v}_2) = \int_\Omega \nabla\mathbf{v}_1 : \nabla\mathbf{v}_2\, d\mathbf{x}, \quad \forall(\mathbf{v}_1,\mathbf{v}_2) \in \mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega),$$

and the trilinear form

$$b(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3) = \int_\Omega (\mathbf{v}_1\nabla)\mathbf{v}_2\cdot\mathbf{v}_3\, d\mathbf{x}, \quad \forall(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3) \in \mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega).$$

Thanks to Hölder inequality, we obtain

$$|b(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)| \quad \leq \quad \|\mathbf{v}_1\|_{\mathbf{L}^3(\Omega)} \, \|\nabla \mathbf{v}_2\| \, \|\mathbf{v}_3\|_{\mathbf{L}^6(\Omega)}, \quad \forall \mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3 \in \mathbf{H}^1(\Omega).$$

Using the generalized Sobolev's inequality, leads to

$$\|\mathbf{v}_1\|_{\mathbf{L}^3(\Omega)} \leq C\|\mathbf{v}_1\|^{\frac{1}{2}}\|\nabla \mathbf{v}_1\|^{\frac{1}{2}} \quad \text{and} \quad \|\mathbf{v}_3\|_{\mathbf{L}^6(\Omega)} \leq C\|\nabla \mathbf{v}_3\|, \quad \text{for } d = 2, 3,$$

where C is a positive constant, and hence

$$|b(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)| \leq C\|\mathbf{v}_1\|^{\frac{1}{2}}\|\nabla \mathbf{v}_1\|^{\frac{1}{2}}\|\nabla \mathbf{v}_2\|\|\nabla \mathbf{v}_3\|. \tag{2.14}$$

By employing integration by parts, the following properties hold true

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) \quad = \quad \frac{1}{2}\int_{\Gamma_b} |\mathbf{v}|^2(\mathbf{u}\cdot\mathbf{n})\, d\zeta, \ \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}(\Omega), \tag{2.15}$$

$$b(\mathbf{v}, \mathbf{v}, \mathbf{v}) \quad = \quad \frac{1}{2}\int_{\Gamma_b} |\mathbf{v}|^2(\mathbf{v}\cdot\mathbf{n})\, d\zeta, \ \forall \mathbf{v} \in \mathbf{V}(\Omega). \tag{2.16}$$

Thanks to [21, Lemma 1.1] we obtain

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{u})| \leq C_\Omega \|\nabla \mathbf{v}\|\|\nabla \mathbf{u}\|^2, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \ \mathbf{u} \in \mathbf{H}_0^1(\Omega), \tag{2.17}$$

where

$$C_\Omega = \begin{cases} \frac{2\sqrt{2}|\Omega|^{1/6}}{3} & \text{if } d = 3 \\ \frac{|\Omega|^{1/2}}{2} & \text{if } d = 2. \end{cases} \tag{2.18}$$

Remark 2.3. *In the stationary case, if $\psi(t, \mathbf{x}) \equiv \psi(\mathbf{x})$ is the solution of the Navier-Stokes problem*

$$\begin{cases} -\nu\Delta\boldsymbol{\psi} + (\boldsymbol{\psi}\cdot\nabla)\boldsymbol{\psi} + \nabla q = \mathbf{f}, \quad \nabla\cdot\boldsymbol{\psi} = 0 & \text{in } \Omega, \\ \boldsymbol{\psi} = 0 & \text{on } \Gamma_l, \\ \boldsymbol{\psi} = \boldsymbol{\psi}_b & \text{on } \Gamma_b, \end{cases} \tag{2.19}$$

with $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, $\boldsymbol{\psi}_b \in V^{1/2}(\Gamma_b)$ and if the smallness condition

$$\|\nabla\boldsymbol{\psi}\| < \frac{\nu}{C_\Omega} \tag{2.20}$$

is satisfied (in view of [21, Theorem 2.1]), then $\psi(\mathbf{x})$ is unique. Moreover $\psi(\mathbf{x})$ belongs to $\mathcal{U}_{ad}$, defined in (2.2).

In the next Section, the control low is built by employing energy a priori estimations.

3 Control building

In the first step a Galerkin basis is built for the space $W(Q)$ defined in (2.12).

3.1 A Galerkin basis for space $W(Q)$

Let $\{\mathbf{z}_j, \lambda_j, j = 1, 2, 3, \cdots\}$ be the eigenfunctions and eigenvalues of the following spectral problem for the Stokes operator :

$$-\Delta \mathbf{z}_j + \nabla p_j = \lambda_j \mathbf{z}_j, \qquad \nabla \cdot \mathbf{z}_j = 0 \quad \text{in } \Omega; \qquad \mathbf{z}_j|_\Gamma = 0. \tag{2.21}$$

As shown in [33], $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty$ as $j \to \infty$, and $\{\mathbf{z}_j\}$ forms an orthonormal basis in $\mathbf{V}_0(\Omega)$:

$$\begin{cases} \langle \mathbf{z}_j, \mathbf{z}_k \rangle = \delta_{jk}, \\ a(\mathbf{z}_j, \mathbf{z}_k) = \lambda_j \delta_{jk}, \quad \forall j, k. \end{cases} \tag{2.22}$$

We assume that the boundary Γ_b is composed of two connected components such that $\Gamma_b = \Gamma_0 \cup \Gamma_1$. Let g_0 such that $\mathrm{g}_0 \in V^{1/2}(\Gamma_0)$ and $\int_{\Gamma_0} \mathrm{g}_0 \cdot \mathbf{n} \neq 0$, we consider the following problem

$$\begin{cases} (a) \quad -\Delta \mathbf{w} + \nabla q = 0, \quad \nabla \cdot \mathbf{w} = 0 & \text{in } \Omega, \\ (b) \quad \mathbf{w} = 0 & \text{on } \Gamma_l, \\ (c) \quad \mathbf{w} = \beta \mathbf{g}_0 & \text{on } \Gamma_0, \\ (d) \quad \mathbf{w} = \mathbf{g}_1 & \text{on } \Gamma_1, \end{cases} \tag{2.23}$$

where g_1 is such that $\mathrm{g}_1 \in V^{1/2}(\Gamma_1)$ with $\mathrm{g}_1 \cdot \mathbf{n} \neq 0$ on Γ_1 and $\beta = -\dfrac{\int_{\Gamma_1} \mathrm{g}_1 \cdot \mathbf{n}\, d\zeta}{\int_{\Gamma_0} \mathrm{g}_0 \cdot \mathbf{n}\, d\zeta}$.

Further, let

$$\mathrm{g} = \begin{cases} \beta \mathrm{g}_0 & \text{on } \Gamma_0 \\ \mathrm{g}_1 & \text{on } \Gamma_1, \end{cases} \tag{2.24}$$

we see by construction that g belongs to $V^{1/2}(\Gamma_b)$ and satisfies $\int_{\Gamma_b} \mathrm{g} \cdot \mathbf{n}\, d\zeta = 0$. Since $\mathrm{w} = \mathrm{g}$ on $\Gamma_b = \Gamma_0 \cup \Gamma_1$, system (2.23) admits a unique solution (w, q) belonging to $\mathbf{H}^1(\Omega) \times L^2_0(\Omega)$ (see [14, Proposition III.4.1]). Moreover, we notice that $\langle \nabla \mathrm{w}, \nabla \mathbf{z}_j \rangle = 0$, for all $j = 1, 2, 3, \cdots$, and the sequence $\mathrm{w},\ \mathbf{z}_1,\ \mathbf{z}_2,\ \mathbf{z}_3, \cdots$, is linearly independent. Consequently, we search for the solution v of (2.4), coupled with (2.7), in

$$W(Q) = \text{span}(\mathbf{w}) \oplus \text{span}(\mathbf{z}_n)_{\{n\in\mathbb{N}^*\}}, \tag{2.25}$$

and v can be expressed as :

$$\mathbf{v} = \alpha\mathbf{w} + \mathbf{z}, \quad \text{with} \quad \mathbf{z} = \sum_{i=1}^{\infty} \theta_i \mathbf{z}_i. \tag{2.26}$$

3.2 The control Building

Multiplying (2.8-a) by v and integrating by parts over Ω leads to

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\|\nabla\mathbf{v}\|^2 + b(\mathbf{v},\mathbf{v},\mathbf{v}) + b(\boldsymbol{\psi},\mathbf{v},\mathbf{v}) + b(\mathbf{v},\boldsymbol{\psi},\mathbf{v}) = \alpha f(\mathbf{v},\alpha). \tag{2.27}$$

Since $\mathbf{v} = \alpha\mathbf{w} + \mathbf{z}$, the control law $f(\mathbf{v},\alpha)$ is built by employing the terms $\|\mathbf{v}\|^2$, $\|\nabla\mathbf{v}\|^2$, $b(\mathbf{v},\boldsymbol{\psi},\mathbf{v})$, $b(\mathbf{v},\mathbf{v},\mathbf{v})$ and $b(\boldsymbol{\psi},\mathbf{v},\mathbf{v})$ in (2.27), which are developed as follows :

$$\|\mathbf{v}\|^2 = \alpha^2\|\mathbf{w}\|^2 + 2\alpha\langle\mathbf{w},\mathbf{z}\rangle + \|\mathbf{z}\|^2, \tag{2.28}$$

$$\|\nabla\mathbf{v}\|^2 = \alpha^2\|\nabla\mathbf{w}\|^2 + \|\nabla\mathbf{z}\|^2, \tag{2.29}$$

$$b(\mathbf{v},\boldsymbol{\psi},\mathbf{v}) = b(\mathbf{z},\boldsymbol{\psi},\mathbf{z}) + \alpha A_{\mathbf{z}} + \alpha^2 B_s, \tag{2.30}$$

where

$$A_{\mathbf{z}} = b(\mathbf{w},\boldsymbol{\psi},\mathbf{z}) + b(\mathbf{z},\boldsymbol{\psi},\mathbf{w}) \quad \text{and} \quad B_s = b(\mathbf{w},\boldsymbol{\psi},\mathbf{w}).$$

Note that the term $\|\nabla\mathbf{v}\|^2$ in (2.27) reduced to (2.29) because $\langle\nabla\mathbf{w},\nabla\mathbf{z}\rangle = 0$.

Further, due to (2.15)-(2.16), the terms $b(\mathbf{v},\mathbf{v},\mathbf{v})$ and $b(\boldsymbol{\psi},\mathbf{v},\mathbf{v})$ are rewritten, respectively, as

$$b(\mathbf{v},\mathbf{v},\mathbf{v}) = \frac{1}{2}\int_{\Gamma_b} |\mathbf{v}|^2(\mathbf{v}\cdot\mathbf{n}) = a_b\alpha^3, \tag{2.31}$$

$$b(\boldsymbol{\psi},\mathbf{v},\mathbf{v}) = \frac{1}{2}\int_{\Gamma_b} |\mathbf{v}|^2(\boldsymbol{\psi}\cdot\mathbf{n}) = b_b\alpha^2, \tag{2.32}$$

where

$$a_b = \frac{1}{2}\int_{\Gamma_b} |\mathbf{g}|^2(\mathbf{g}\cdot\mathbf{n}) \quad \text{and} \quad b_b = \frac{1}{2}\int_{\Gamma_b} |\mathbf{g}|^2(\boldsymbol{\psi}\cdot\mathbf{n}),$$

and the functions b_b, A and B are time dependent.

Substituting (2.29)-(2.30) in (2.27), yields

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\alpha^2\|\nabla\mathbf{w}\|^2 + \nu\|\nabla\mathbf{z}\|^2 + b(\mathbf{z},\boldsymbol{\psi},\mathbf{z}) + \alpha S(\mathbf{z},\alpha) = \alpha f(\mathbf{v},\alpha), \tag{2.33}$$

where

$$S(\mathbf{z},\alpha) = A_{\mathbf{z}} + B_s\alpha + a_b\alpha^2 + b_b\alpha.$$

The control law is now defined as

$$f(\mathbf{v},\alpha) = S(\mathbf{z},\alpha) - \lambda_\nu\left(2\langle\mathbf{w},\mathbf{z}\rangle + \alpha\|\mathbf{w}\|^2\right) - K\alpha\|\mathbf{v}\|^2, \tag{2.34}$$

where the positive constants K and λ_ν will be defined later. Note that in (2.34), the term $2\langle\mathbf{w},\mathbf{z}\rangle + \alpha\|\mathbf{w}\|^2$ is a part of $\|\mathbf{v}\|^2$ in (2.28) while the term $-K\alpha\|\mathbf{v}\|^2$ is introduced in order to limit the size of the control, for an appropriate choice of K.

4 Stability Result

We first establish the a priori estimates for the extended Navier-Stokes system.

4.1 A priori estimates

Multiplying (2.34) by α, substituting in (2.33) and using (2.28) leads to

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\alpha^2\|\nabla\mathbf{w}\|^2 + \nu\|\nabla\mathbf{z}\|^2 + b(\mathbf{z},\boldsymbol{\psi},\mathbf{z}) = -\lambda_\nu\left(\|\mathbf{v}\|^2 - \|\mathbf{z}\|^2\right) - K\|\mathbf{v}\|^2\alpha^2. \tag{2.35}$$

We obtain from (2.17)

$$|b(\mathbf{z},\boldsymbol{\psi},\mathbf{z})| \le C_\Omega\|\nabla\boldsymbol{\psi}\|\|\nabla\mathbf{z}\|^2 \le C_\Omega \sup_{t\le T}\|\nabla\boldsymbol{\psi}(t,\mathbf{x})\|\|\nabla\mathbf{z}\|^2,$$

and due to (2.35) we have

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\alpha^2\|\nabla\mathbf{w}\|^2 + \beta_\nu\|\nabla\mathbf{z}\|^2 \le -\lambda_\nu\left(\|\mathbf{v}\|^2 - \|\mathbf{z}\|^2\right) - K\|\mathbf{v}\|^2\alpha^2, \tag{2.36}$$

where $\beta_\nu = \nu - C_\Omega \sup_{t\le T}\|\nabla\boldsymbol{\psi}(t,\mathbf{x})\|$. Moreover, since

$$\lambda_1\|\mathbf{z}\|^2 = \lambda_1\sum_{i=1}^{\infty}\theta_i^2 \le \sum_{i=1}^{\infty}\lambda_i\theta_i^2 = \|\nabla\mathbf{z}\|^2,$$

and taking $\lambda_\nu = \lambda_1\beta_\nu$ in (2.36) yields

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\alpha^2\|\nabla\mathbf{w}\|^2 + \left(\lambda_\nu + K\alpha^2\right)\|\mathbf{v}\|^2 \le 0,$$

namely

$$\frac{d}{dt}\|\mathbf{v}\|^2 + 2\left(\lambda_\nu + K\alpha^2\right)\|\mathbf{v}\|^2 \leq 0. \tag{2.37}$$

Multiplying (2.37) by $e^{2\sigma(t)}$ where

$$\sigma(t) = \lambda_\nu t + K\int_0^t \alpha^2(s)ds \tag{2.38}$$

leads to

$$\frac{d}{dt}(\|\mathbf{v}\|^2 e^{2\sigma(t)}) \leq 0. \tag{2.39}$$

By integrating (2.39) from 0 to t we obtain the first a priori estimate

$$\|\mathbf{v}\| \leq \|\mathbf{v}(0)\|e^{-\sigma(t)}. \tag{2.40}$$

Note that due to (2.2-a), we have $\beta_\nu = \nu - C_\Omega \sup_{t\leq T}\|\nabla\boldsymbol{\psi}(t,\mathbf{x})\| > 0$, hence $\lambda_\nu = \lambda_1\beta_\nu > 0$. We then obtain from (2.36)

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \beta_\nu\alpha^2\|\nabla\mathbf{w}\|^2 + \beta_\nu\|\nabla\mathbf{z}\|^2 \leq \lambda_\nu\|\mathbf{z}\|^2, \tag{2.41}$$

and using (2.29), we deduce

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \beta_\nu\|\nabla\mathbf{v}\|^2 \leq \lambda_\nu\|\mathbf{z}\|^2. \tag{2.42}$$

Let us estimate the term in the right hand side of (2.42). Since

$$\|\mathbf{z}\|^2 = \|\mathbf{v} - \alpha\mathbf{w}\|^2 \leq 2\|\mathbf{v}\|^2 + 2\alpha^2\|\mathbf{w}\|^2,$$

using Lemma 2.2, we obtain

$$\lambda_\nu\|\mathbf{z}\|^2 \leq M_1\|\mathbf{v}\|^2, \tag{2.43}$$

where $M_1 = 2\lambda_\nu\left(1 + C_b^2\|\mathbf{w}\|^2\right)$. Further, employing (2.43) in (2.42) leads to

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \beta_\nu\|\nabla\mathbf{v}\|^2 \leq M_1\|\mathbf{v}\|^2. \tag{2.44}$$

Integrating (2.44) from 0 to t, yields

$$\|\mathbf{v}\|^2 + 2\beta_\nu\int_0^t \|\nabla\mathbf{v}\|^2 ds \quad \leq \quad \|\mathbf{v}(0)\|^2 + 2M_1\int_0^t \|\mathbf{v}\|^2 ds, \tag{2.45}$$

using the inequality $e^{-\sigma(t)} \leq e^{-\lambda_\nu t}$ obtained from (2.38), and integrating (2.40) from 0 to t, we obtain

$$\begin{aligned} \|\mathbf{v}\|^2 + 2\beta_\nu \int_0^t \|\nabla \mathbf{v}\|^2 ds &\leq \left(1 + \frac{M_1}{\lambda_\nu} - \frac{M_1}{\lambda_\nu} e^{-2\lambda_\nu t}\right) \|\mathbf{v}(0)\|^2 \\ &\leq \left(1 + \frac{M_1}{\lambda_\nu}\right) \|\mathbf{v}(0)\|^2 = \left(3 + 2C_b^2 \|\mathbf{w}\|^2\right) \|\mathbf{v}(0)\|^2. \end{aligned}$$

Therefore, we obtain the a priori estimate

$$\int_0^t \|\nabla \mathbf{v}\|^2 ds \leq \left(\frac{3 + 2C_b^2 \|\mathbf{w}\|^2}{2\beta_\nu}\right) \|\mathbf{v}(0)\|^2. \tag{2.46}$$

4.2 The variational formulation

We now consider the variational formulation for the extended Navier-Stokes system.

Definition 4.1. *Let $T > 0$ be an arbitrary real number and $\mathbf{v}_0 \in \mathbf{H}(\Omega)$, we shall say that $(\mathbf{v}, \alpha)$ is a weak solution of (2.8) on $[0, T)$ if*

- $\mathbf{v} \in L^\infty(0, T; \mathbf{H}(\Omega)) \cap L^2(0, T; \mathbf{V}(\Omega))$,
- $\exists \alpha \in L^\infty(0, T)$ *such that* $\mathbf{v} = \alpha \mathbf{g}$ *on* Γ_b,

$$\begin{cases} (a) & \langle d_t \mathbf{v}, \widetilde{\mathbf{v}} \rangle + \nu a(\mathbf{v}, \widetilde{\mathbf{v}}) + b(\mathbf{v}, \boldsymbol{\psi}, \widetilde{\mathbf{v}}) + b(\boldsymbol{\psi}, \mathbf{v}, \widetilde{\mathbf{v}}) + b(\mathbf{v}, \mathbf{v}, \widetilde{\mathbf{v}}) = \widetilde{\alpha} f(\mathbf{v}, \alpha), \\ (b) & \left(\displaystyle\int_\Omega \mathbf{v} \cdot \tilde{\mathbf{v}}\, d\mathbf{x}\right)(0) = \displaystyle\int_\Omega \mathbf{v}_0 \cdot \tilde{\mathbf{v}}\, d\mathbf{x}, \end{cases} \tag{2.47}$$

$\forall (\widetilde{\mathbf{v}}, \widetilde{\alpha}) \in W(Q)$.

Note that the initial condition (2.47-b) makes sense because for any solution $\mathbf{v}$ of (2.47-a), function $t \to \int_\Omega \mathbf{v}(t) \cdot \tilde{\mathbf{v}}\, d\mathbf{x}$ is continuous (see [14] Corollaire II.4.2).

Theorem 4.2. *Assume that the initial condition $\mathbf{v}_0$ and the profile $\mathbf{g}$ satisfy*

$$\mathbf{v}_0 \in \mathbf{H}(\Omega), \quad (\mathbf{v}_0 \cdot \mathbf{n})\, \mathbf{n} \in \mathbf{H}^{1/2}(\Gamma_b), \tag{2.48}$$

$$\mathbf{g} \in \mathbf{V}^{1/2}(\Gamma_b) \quad \text{and} \quad \alpha_0\, \mathbf{g} \cdot \mathbf{n} = \mathbf{v}_0 \cdot \mathbf{n} \ \text{ on } \ \Gamma_b \ \text{ with } \ \mathbf{g} \cdot \mathbf{n} \neq 0,\ \alpha_0 \in \mathbb{R}. \tag{2.49}$$

For arbitrary initial data $\mathbf{v}_0$ satisfying (2.48), there exists a solution $(\mathbf{v}, \alpha)$ in the sense of definition 4.1, and a distribution p on Q such that (2.8) holds. Moreover, function $\mathbf{v}$

satisfies the following estimates :

$$\|\mathbf{v}(t)\| \leq \|\mathbf{v}_0\| e^{-\sigma(t)}, \ \forall t > 0, \tag{2.50}$$

$$\int_0^T \|\nabla \mathbf{v}(t)\|^2 dt \leq C\|\mathbf{v}_0\|^2, \tag{2.51}$$

where $C > 0$ is a constant and for a fixed $K > 0$, function $\sigma(t)$ is defined as :

$$\sigma(t) = \lambda_1 \beta_\nu t + K \int_0^t \alpha^2(s) ds. \tag{2.52}$$

Remark 4.3. *In (2.52), the positive constant λ_1 is the smallest eigenvalue of (2.21) and thanks to (2.2-a), the constant $\beta_\nu = \nu - C_\Omega \sup_{t \leq T} \|\nabla \psi(t, \mathbf{x})\|$ is a positive real number. Further, the rate of decrease $\sigma(t) > 0$ depends on the control α.*

Remark 4.4. *With the condition $\beta_\nu > 0$, the equilibrium state ψ in (2.1) is naturally stable in the sense that the system (2.47) stabilizes by itself when α is identically zero. This explains why the choice of the initial perturbation $\mathbf{v}_0$, in Theorem 4.2, is arbitrary. However, as shown in Proposition 3.1, the control α is not identically zero as soon as the initial perturbation $\mathbf{v}_0$ and the profile $\mathbf{g}$ satisfy (2.48)-(2.49) with $\mathbf{v}_0 \cdot \mathbf{n} \neq 0$. The theoretical case $\mathbf{v}_0 \cdot \mathbf{n} = 0$ remains an open question.*

Proof. We first proof the existence of a weak solution $(\mathbf{v}, \alpha)$ and secondly, the existence of the pressure.

4.3 Existence of weak solution

The proof of the existence follows a standard procedure. In a first step a sequence of approximate solutions using a Galerkin method is built. A compactness result from [26] allows us to pass to the limit in the system satisfied by the approximated solutions.

4.3.1 The Galerkin Method

Let $m \in \mathbb{N}^*$, we define the space

$$W_m = \text{span}(\mathbf{w}) \oplus \text{span}(\mathbf{z}_i)_{\{1 \leq i \leq m\}}$$

and we express $\mathbf{v}_m \in W_m$ as :

$$\mathbf{v}_m = \sum_{i=0}^{m} \alpha_{im} \mathbf{w}_i,$$

where $\mathbf{w}_0 = \mathbf{w}$ and $\mathbf{w}_i = \mathbf{z}_i$ for $i = 1, 2, 3 \cdots, m$.

Consider the following finite-dimensional problem

$$
\begin{cases}
(a) \quad \langle d_t \mathbf{v}_m, \mathbf{w}_j \rangle + \nu a(\mathbf{v}_m, \mathbf{w}_j) + b(\mathbf{v}_m, \boldsymbol{\psi}, \mathbf{w}_j) + b(\boldsymbol{\psi}, \mathbf{v}_m, \mathbf{w}_j) + b(\mathbf{v}_m, \mathbf{v}_m, \mathbf{w}_j) \\
\qquad = \delta_{0j} f(\mathbf{v}_m, \alpha_{0m}), \\
(b) \quad \langle \mathbf{v}_m(0) - \mathbf{v}_0, \mathbf{w}_j \rangle = 0, \text{ for } j = 0, 1, 2, \cdots, m,
\end{cases}
\tag{2.53}
$$

where δ_{0j} is the Kronecker symbol and

$$
f(\mathbf{v}_m, \alpha_{0m}) = A_{\mathbf{z}_m} + B_s \alpha_{0m} + a_b \alpha_{0m}^2 + b_b \alpha_{0m} - 2\lambda_\nu \langle \mathbf{w}, \mathbf{z}_m \rangle - \big(\lambda_\nu \|\mathbf{w}\|^2 + K\|\mathbf{v}_m\|^2\big)\alpha_{0m}, \tag{2.54}
$$

is the control law, with $\mathbf{z}_m = \sum_{i=1}^m \alpha_{im} \mathbf{w}_i$ and $A_{\mathbf{z}_m} = b(\mathbf{w}, \boldsymbol{\psi}, \mathbf{z}_m) + b(\mathbf{z}_m, \boldsymbol{\psi}, \mathbf{w})$.

Recall that α_{0m} is a priori unknown and thanks to (2.54) it satisfies a nonlinear feedback law leading to search for $\alpha_{0m}(\mathbf{v}_m)$. Because (2.54) is independent of x, $\alpha_{0m}(\mathbf{v}_m)$ is a function of t only. For the sake of simplicity, $\alpha_{0m}(\mathbf{v}_m)$ is written α_{0m} in the sequel.

Lemma 4.5. *The discrete problem* (2.53) *has a unique solution* $\mathbf{v}_m \in W^{1,\infty}(0,T;W_m)$. *Moreover the solution satisfies :*

$$
\|\mathbf{v}_m\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{v}_m\|_{L^2(0,T;\mathbf{H}^1(\Omega))} \le C, \tag{2.55}
$$

where C is a positive constant independent of m.

Proof. We rewrite (2.53) in terms of the unknown α_{im}, $i = 0, 1, 2 \cdots m$, and we obtain

$$
\begin{cases}
\displaystyle\sum_{i=0}^m \frac{d\alpha_{im}}{dt} \langle \mathbf{w}_i, \mathbf{w}_j \rangle + \sum_{i=0}^m \alpha_{im} \big(\nu\, a(\mathbf{w}_i, \mathbf{w}_j) + b(\boldsymbol{\psi}, \mathbf{w}_i, \mathbf{w}_j) + b(\mathbf{w}_i, \boldsymbol{\psi}, \mathbf{w}_j)\big) \\
\qquad\qquad \displaystyle + \sum_{i,k=0}^m \alpha_{km} \alpha_{im} b(\mathbf{w}_i, \mathbf{w}_k, \mathbf{w}_j) = \delta_{0j} f(\mathbf{v}_m, \alpha_{0m}), \\
\displaystyle\sum_{i=0}^m \alpha_{im}(0) \langle \mathbf{w}_i, \mathbf{w}_j \rangle = \langle \mathbf{v}_0, \mathbf{w}_j \rangle, \text{ for } j = 0, 1, 2 \cdots, m.
\end{cases}
\tag{2.56}
$$

Since the mass matrix with entries $\langle \mathbf{w}_i, \mathbf{w}_j \rangle$ $(0 \le i, j \le m)$ is nonsingular, (2.56) reduces to a nonlinear system with constant coefficients

$$
\begin{cases}
\displaystyle\frac{d\alpha_{im}}{dt} + \sum_{j=0}^m \alpha_{jm} X_{ij} + \sum_{j,k=0}^m \alpha_{km} \alpha_{jm} Y_{ijk} = \sum_{j=0}^m \delta_{0j} f(\mathbf{v}_m, \alpha_{0m}) Z_{ij}, \\
\displaystyle\alpha_{im}(0) = \sum_{j=0}^m \langle \mathbf{v}_0, \mathbf{w}_j \rangle Z_{ij},
\end{cases}
\tag{2.57}
$$

where $X_{ij}, Y_{ijk}, Z_{ij}, \in \mathbb{R}$. Then, there exists T_m $(0 < T_m \le T)$ such that the nonlinear differential system (2.57) has a maximal solution defined on some interval $[0, T_m]$. In order to show that T_m is independent of m, it is sufficient to verify the boundedness of the L^2-norm of $\mathbf{v}_m$ independently of m.

Multiplying (2.53-b) by $\alpha_{jm}(0)$, and summing for $j = 0, 1, 2, \cdots, m$, lead to

$$\int_\Omega |\mathbf{v}_m(0)|^2 = \int_\Omega \mathbf{v}_0 \cdot \mathbf{v}_m(0) \leq \frac{1}{2}\int_\Omega |\mathbf{v}_0|^2 + \frac{1}{2}\int_\Omega |\mathbf{v}_m(0)|^2,$$

and then

$$\|\mathbf{v}_m(0)\|^2 \leq \|\mathbf{v}_0\|^2. \tag{2.58}$$

Following the same procedure as for the derivation of the a priori estimates (2.40) and (2.46), and using (2.58) yields

$$\begin{cases} (a) \quad \|\mathbf{v}_m\| \leq \|\mathbf{v}_0\|\, e^{-\sigma(t)}, \\ (b) \quad \displaystyle\int_0^T \|\nabla \mathbf{v}_m\|^2 dt \leq C\|\mathbf{v}_0\|^2. \end{cases} \tag{2.59}$$

If $T_m < T$, then $\|\mathbf{v}_m\|$ should tend to $+\infty$ as $t \to T_m$ because of the explosion criteria. However, this does not happen since $\|\mathbf{v}_m\|$ is bounded independently of m in (2.59-a), and therefore $T_m = T$. □

A consequence of the a priori estimates (2.59) is that $(\mathbf{v}_m)_m$ is bounded in $L^2(0, T; \mathbf{V}(\Omega))$ and $L^\infty(0, T; \mathbf{H}(\Omega))$. Therefore, for a subsequence of $\mathbf{v}_m$ (still denoted by $\mathbf{v}_m$), the estimates in (2.59) yield the following weak convergences as m tends to ∞ :

$$\begin{cases} \mathbf{v}_m \rightharpoonup \mathbf{v} \text{ weakly in } L^2(0, T; \mathbf{V}(\Omega)), \\ \mathbf{v}_m \rightharpoonup \mathbf{v} \text{ weakly* in } L^\infty(0, T; \mathbf{H}(\Omega)). \end{cases} \tag{2.60}$$

Nevertheless, the convergences in (2.60) are not sufficient to pass to the limit in the weak formulation (2.53), because of the presence of the convection term. Consequently, we need to obtain additional bounds in order to utilize the compactness theory on the sequence of approximated solution $\mathbf{v}_m$.

4.3.2 Additional bounds

As in [26], let us assume that B_0, B and B_1 are three Hilbert spaces such that $B_0 \subset B \subset B_1$. If a function $\mathbf{v}$ is such that $\mathbf{v} : \mathbb{R} \to B_1$, we denote by $\widehat{\mathbf{v}}$ its Fourier transform

$$\widehat{\mathbf{v}}(\tau) = \int_{-\infty}^{+\infty} e^{-2i\pi t\tau}\mathbf{v}(t)dt.$$

Let us recall the following identity about the Fourier transform of differential operators :

$$\widehat{D_t^\gamma \mathbf{v}(\tau)} = (2i\pi\tau)^\gamma \widehat{\mathbf{v}}(\tau),$$

for a given $\gamma > 0$, and let us define the space

$$H^\gamma(\mathbb{R}; B_0, B_1) = \{\mathbf{u} \in L^2(\mathbb{R}, B_0), D_t^\gamma \mathbf{u} \in L^2(\mathbb{R}, B_1)\}.$$

The space $H^\gamma(\mathbb{R}; B_0, B_1)$ is endowed with the norm

$$\|\mathbf{v}\|_{H^\gamma(\mathbb{R};B_0,B_1)} = (\|\mathbf{v}\|^2_{L^2(\mathbb{R};B_0)} + \||\tau|^\gamma \widehat{\mathbf{v}}\|^2_{L^2(\mathbb{R};B_1)})^{\frac{1}{2}}.$$

We also define $H^\gamma(0,T;B_0,B_1)$, as the space of functions obtained by restriction to $[0,T]$ of functions of $H^\gamma(\mathbb{R};B_0,B_1)$. Further, we recall the following result [26] :

Lemma 4.6. *Let B_0, B and B_1 be three Hilbert spaces such that $B_0 \subset B \subset B_1$ and B_0 is compactly embedded in B. Then for all $\gamma > 0$, the injection $H^\gamma(0,T;B_0,B_1) \to L^2(0,T;B)$ is compact.*

For small enough ε, Lemma 4.6 is used later with

$$B_0 = \mathbf{V}(\Omega), \qquad B = \mathbf{H}(\Omega), \qquad B_1 = \mathbf{H}(\Omega), \qquad \gamma = \frac{1}{4} - \epsilon.$$

The main result of the present section is obtained by using the following Lemma :

Lemma 4.7. *The sequence $\mathbf{v}_m$ is bounded in $H^\gamma(0,T;\mathbf{V}(\Omega),\mathbf{H}(\Omega))$ for $0 \leq \gamma \leq \frac{1}{4} - \epsilon$.*

Proof. We denote by $\bar{\mathbf{v}}_m$ the extension of $\mathbf{v}_m$ by zero for $t < 0$ and $t > T$, and $\widehat{\mathbf{v}}_m$ the Fourier transform of $\bar{\mathbf{v}}_m$ with respect to time. Since $\bar{\mathbf{v}}_m$ has two discontinuities at 0 and T, in the distributional sense, the derivative of $\bar{\mathbf{v}}_m$ is expressed as

$$\frac{d}{dt}\bar{\mathbf{v}}_m = \bar{\mathbf{u}}_m + \mathbf{v}_m(0)\delta_0 - \mathbf{v}_m(T)\delta_T, \tag{2.61}$$

where δ_0, δ_T are Dirac distributions at 0 and T, respectively, and

$$\bar{\mathbf{u}}_m = \mathbf{v}'_m \text{ which denotes the derivative of } \mathbf{v}_m \text{ on } [0,T].$$

The Fourier transformation of (2.61) gives

$$2i\pi\tau\widehat{\mathbf{v}}_m(\tau) = \widehat{\mathbf{u}}_m(\tau) + \mathbf{v}_m(0) - \mathbf{v}_m(T)e^{-2i\pi\tau T},$$

where $\widehat{\mathbf{v}}_m$ and $\widehat{\mathbf{u}}_m$ denote the Fourier transforms of $\bar{\mathbf{v}}_m$ and $\bar{\mathbf{u}}_m$ respectively. Since $\mathbf{v}_m$ is uniformly bounded in $L^2(0,T,\mathbf{V}(\Omega))$, it remains to prove that

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma}\|\widehat{\mathbf{v}}_m(\tau)\|d\tau \leq C. \tag{2.62}$$

For all $(\widetilde{\mathbf{v}},\widetilde{\alpha}) \in W_m$, we have that $\bar{\mathbf{v}}_m$ satisfies

$$\begin{aligned}\int_\Omega \frac{\partial \bar{\mathbf{v}}_m}{\partial t}\cdot\widetilde{\mathbf{v}}\; d\mathbf{x} + \nu\int_\Omega \nabla\bar{\mathbf{v}}_m : \nabla\widetilde{\mathbf{v}}\; d\mathbf{x} + \int_\Omega G_m\cdot\widetilde{\mathbf{v}}\; d\mathbf{x} + \int_\Omega G_m^0\cdot\widetilde{\mathbf{v}}\; d\mathbf{x} + \int_\Omega G_m^1\cdot\widetilde{\mathbf{v}}\; d\mathbf{x}\\ = -\int_\Omega \bar{\mathbf{v}}_m(T)\cdot\widetilde{\mathbf{v}}\delta_T\; d\mathbf{x} + \int_\Omega \bar{\mathbf{v}}_m(0)\cdot\widetilde{\mathbf{v}}\delta_0\; d\mathbf{x} + \widetilde{\alpha}H_m,\end{aligned} \tag{2.63}$$

where $G_m = (\bar{\mathbf{v}}_m\cdot\nabla)\bar{\mathbf{v}}_m$, $G_m^0 = (\bar{\mathbf{v}}_m\cdot\nabla)\boldsymbol{\psi}$, $G_m^1 = (\boldsymbol{\psi}\cdot\nabla)\bar{\mathbf{v}}_m$ and $H_m = f(\bar{\mathbf{v}}_m,\bar{\alpha}_{0m})$. We apply the Fourier transform to (2.63) and take $(\widehat{\mathbf{v}}_m,\widehat{\alpha}_{0m})$ as a test function, which leads to

$$\begin{aligned}2i\pi\tau\int_\Omega |\widehat{\mathbf{v}}_m(\tau)|^2\; d\mathbf{x} + \nu\int_\Omega \nabla\widehat{\mathbf{v}}_m(\tau) : \nabla\widehat{\mathbf{v}}_m(\tau)\; d\mathbf{x} + \int_\Omega \widehat{G}_m(\tau)\cdot\widehat{\mathbf{v}}_m(\tau)\; d\mathbf{x}\\ + \int_\Omega \widehat{G}_m^0(\tau)\cdot\widehat{\mathbf{v}}_m(\tau)\; d\mathbf{x} + \int_\Omega \widehat{G}_m^1(\tau)\cdot\widehat{\mathbf{v}}_m(\tau)\; d\mathbf{x} = \int_\Omega \bar{\mathbf{v}}_m(0)\cdot\widehat{\mathbf{v}}_m(\tau)\; d\mathbf{x}\\ - \int_\Omega \bar{\mathbf{v}}_m(T)\cdot\widehat{\mathbf{v}}_m(\tau)e^{-2i\pi\tau T}\; d\mathbf{x} + \widehat{\alpha}_{0m}\widehat{H}_m.\end{aligned} \tag{2.64}$$

where $\widehat{G}_m$, $\widehat{G}_m^0$, $\widehat{G}_m^1$ and $\widehat{H}_m$ are respectively the Fourier transform with respect to time of G_m, G_m^0, G_m^1 and H_m. Taking the Fourier transform of (2.54) and multiplying it by $\widehat{\alpha}_{0m}$, yields

$$\widehat{\alpha}_{0m}\widehat{H}_m = \widehat{\alpha}_{0m}\widehat{F}_m + b_b(\widehat{\alpha}_{0m})^2 - 2\widehat{\alpha}_{0m}\lambda_\nu\langle\mathbf{w},\widehat{\mathbf{z}}_m\rangle - \lambda_\nu\|\mathbf{w}\|^2(\widehat{\alpha}_{0m})^2, \tag{2.65}$$

where $\widehat{F}_m$ is the Fourier transform of F_m with respect to time, with

$$F_m = A_{\bar{\mathbf{z}}_m} + B_s\bar{\alpha}_{0m} + a_b\bar{\alpha}_{0m}^2 - K\bar{\alpha}_{0m}\|\bar{\mathbf{v}}_m\|^2. \tag{2.66}$$

By rewriting the two last terms of (2.65), we obtain

$$\widehat{\alpha}_{0m}\widehat{H}_m = \widehat{\alpha}_{0m}\widehat{F}_m + b_b(\widehat{\alpha}_{0m})^2 - \lambda_\nu\big(\|\widehat{\mathbf{v}}_m\|^2 - \|\widehat{\mathbf{z}}_m\|^2\big). \tag{2.67}$$

Thanks to Lemma 2.2, we have $|\widehat{\alpha}_{0m}(\tau)| \leq C_b\|\widehat{\mathbf{v}}_m(\tau)\|$, and substituting (2.67) in (2.64), and taking the imaginary part of (2.64) leads to

$$\begin{aligned}|\tau|\|\widehat{\mathbf{v}}_m(\tau)\|^2 \leq C\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{V}(\Omega)}\Big(\|\widehat{G}_m(\tau)\|_{\mathbf{V}'(\Omega)} + \|\widehat{G}_m^0(\tau)\|_{\mathbf{V}'(\Omega)} + \|\widehat{G}_m^1(\tau)\|_{\mathbf{V}'(\Omega)}\Big)\\ + C\,\|\widehat{\mathbf{v}}_m(\tau)\|\big(|\widehat{F}_m| + \|\bar{\mathbf{v}}_m(T)\| + \|\bar{\mathbf{v}}_m(0)\|\big).\end{aligned} \tag{2.68}$$

Note that in the sequel, C stands for different positive constants.

We now prove that each term lying in the right hand side of (2.68) is bounded.

Firstly, by using (2.14) and the definition of G_m, we have

$$|\langle G_m, \mathbf{u}\rangle| = |b(\bar{\mathbf{v}}_m, \bar{\mathbf{v}}_m, \mathbf{u})| \leq C\|\mathbf{v}_m\|^{\frac{1}{2}}\|\nabla \mathbf{v}_m\|^{\frac{1}{2}}\|\nabla \mathbf{v}_m\|\|\nabla \mathbf{u}\|, \ \forall \mathbf{u} \in \mathbf{V}(\Omega),$$

and since $\mathbf{v}_m = 0$ on Γ_l which is a part of the domain boundary, due to the Poincaré inequality, there exists a constant C such that $\|\mathbf{v}_m\| \leq C\|\nabla \mathbf{v}_m\|$, hence

$$\|G_m\|_{\mathbf{V}'(\Omega)} \leq C\|\mathbf{v}_m\|^2_{\mathbf{H}^1(\Omega)}.$$

Secondly, by employing (2.14) and the definition of $G^s_m, \ s = 0, 1$, lead to

$$|\langle G^0_m, \mathbf{u}\rangle| = |b(\bar{\mathbf{v}}_m, \boldsymbol{\psi}, \mathbf{u})| \leq C\|\mathbf{v}_m\|^{\frac{1}{2}}\|\nabla \mathbf{v}_m\|^{\frac{1}{2}}\|\nabla \boldsymbol{\psi}\|\|\nabla \mathbf{u}\|, \ \forall \mathbf{u} \in \mathbf{V}(\Omega), \tag{2.69}$$

$$|\langle G^1_m, \mathbf{u}\rangle| = |b(\boldsymbol{\psi}, \bar{\mathbf{v}}_m, \mathbf{u})| \leq C\|\boldsymbol{\psi}\|^{\frac{1}{2}}\|\nabla \boldsymbol{\psi}\|^{\frac{1}{2}}\|\nabla \mathbf{v}_m\|\|\nabla \mathbf{u}\|, \ \forall \mathbf{u} \in \mathbf{V}(\Omega). \tag{2.70}$$

Further, since $\boldsymbol{\psi} = 0$ on Γ_l, we deduce from (2.69)-(2.70)

$$|\langle G^s_m, \mathbf{u}\rangle| \leq C\|\nabla \mathbf{v}_m\|\|\nabla \mathbf{u}\|, \ \forall \mathbf{u} \in \mathbf{V}(\Omega), \ s = 0, 1,$$

and hence

$$\|G^s_m\|_{\mathbf{V}'(\Omega)} \leq C\|\mathbf{v}_m\|_{\mathbf{H}^1(\Omega)}, \quad s = 0, 1.$$

Consequently, thanks to the energy estimate (2.59) satisfied by $\mathbf{v}_m$, G_m and G^s_m remain bounded in $L^1(\mathbb{R}; \mathbf{V}'(\Omega))$ and the functions $\widehat{G}_m, \ \widehat{G}^s_m$ are bounded in $L^\infty(\mathbb{R}; \mathbf{V}'(\Omega))$ i.e.

$$\sup_{\tau \in \mathbb{R}} (\|\widehat{G}_m(\tau)\|_{\mathbf{V}'(\Omega)} + \|\widehat{G}^0_m(\tau)\|_{\mathbf{V}'(\Omega)} + \|\widehat{G}^1_m(\tau)\|_{\mathbf{V}'(\Omega)}) \leq C.$$

We now show that the last three terms in the right hand side of (2.68) are bounded. Thanks to the energy estimate (2.59-a) satisfied by $\mathbf{v}_m$, we have $\|\mathbf{v}_m(T)\| \leq C$ and $\|\mathbf{v}_m(0)\| \leq C$. Moreover, since $\boldsymbol{\psi} \in \mathcal{U}_{ad}$, thanks to Lemma 2.2 and (2.59-a), we show that each term of F_m defined in (2.66) is bounded in $L^1(\mathbb{R})$, hence F_m is bounded in $L^1(\mathbb{R})$. Therefore, $\widehat{F}_m$ is bounded in $L^\infty(\mathbb{R})$ i.e.

$$\sup_{\tau \in \mathbb{R}} |\widehat{F}_m| \leq C.$$

Inequality (2.68) finally reduces to

$$|\tau|\|\widehat{\mathbf{v}}_m(\tau)\|^2 \leq C(\|\widehat{\mathbf{v}}_m(\tau)\| + \widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}) \leq C\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)},$$

where C stands for different positive constants.

For $0 < \gamma < \frac{1}{4}$, we now estimate the norm

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \|\widehat{\mathbf{v}}_m(\tau)\|^2 d\tau. \tag{2.71}$$

Note that, (see [26])

$$|\tau|^{2\gamma} \leq c(\gamma) \frac{1 + |\tau|}{1 + |\tau|^{1-2\gamma}}, \ \forall \tau \in \mathbb{R}.$$

Consequently, we deduce

$$\begin{aligned}
\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \|\widehat{\mathbf{v}}_m(\tau)\|^2 d\tau &\leq c(\gamma) \int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|^2}{1 + |\tau|^{1-2\gamma}} d\tau + c(\gamma) \int_{-\infty}^{+\infty} \frac{|\tau| \|\widehat{\mathbf{v}}_m(\tau)\|^2}{1 + |\tau|^{1-2\gamma}} d\tau \\
&\leq c_3(\gamma) \int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|^2_{\mathbf{H}^1(\Omega)}}{1 + |\tau|^{1-2\gamma}} d\tau + c_4(\gamma) \int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}}{1 + |\tau|^{1-2\gamma}} d\tau \\
&\leq c_3(\gamma) \int_{-\infty}^{+\infty} \|\widehat{\mathbf{v}}_m(\tau)\|^2_{\mathbf{H}^1(\Omega)} d\tau + c_4(\gamma) \int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}}{1 + |\tau|^{1-2\gamma}} d\tau. \quad (2.72)
\end{aligned}$$

The last integral in the right hand side of (2.72) satisfies

$$\int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}}{1 + |\tau|^{1-2\gamma}} d\tau \leq \left(\int_{-\infty}^{+\infty} \frac{1}{(1 + |\tau|^{1-2\gamma})^2} d\tau \right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} \|\widehat{\mathbf{v}}_m(\tau)\|^2_{\mathbf{H}^1(\Omega)} d\tau \right)^{\frac{1}{2}}, \tag{2.73}$$

and the first integral in the right hand side of (2.73) is convergent for any $0 < \gamma < \frac{1}{4}$. On the other hand, using the Parseval equality leads to

$$\int_{-\infty}^{+\infty} \|\widehat{\mathbf{v}}_m(\tau)\|^2_{\mathbf{H}^1(\Omega)} d\tau = \int_0^T \|\mathbf{v}_m(t)\|^2_{\mathbf{H}^1(\Omega)} dt \leq C.$$

Then, the sequence $\mathbf{v}_m$ is bounded in $H^\gamma(0, T; \mathbf{V}(\Omega), \mathbf{H}(\Omega))$, for $0 \leq \gamma \leq \frac{1}{4} - \epsilon$. $\square$

Finally, by applying Lemmas 4.6 and 4.7, there exists a subsequence of $\mathbf{v}_m$ which converges strongly in $L^2(0, T, \mathbf{H}(\Omega))$.

4.3.3 Passage to the limit

The compactness result obtained in the previous section implies the following strong convergence (at least for a subsequence of $\mathbf{v}_m$ still denoted $\mathbf{v}_m$)

$$\mathbf{v}_m \to \mathbf{v} \text{ strongly in } L^2(0, T; \mathbf{L}^2(\Omega)).$$

Using the above strong convergence result and (2.60) enable us to pass to the limit in the weak formulation. Note that the weak formulation is obtained by multiplying (2.53) by $\varphi \in \mathcal{D}(]0,T[)$, and using integration by parts with respect to time leads to

$$\begin{aligned} -\int_0^T \int_\Omega \mathbf{v}_m \cdot \widetilde{\mathbf{v}}_j \varphi'(t)\, d\mathbf{x}dt + \nu \int_0^T \int_\Omega \nabla \mathbf{v}_m : \nabla \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt + \int_0^T \int_\Omega (\mathbf{v}_m \cdot \nabla \mathbf{v}_m) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt \\ + \int_0^T \int_\Omega (\mathbf{v}_m \cdot \nabla \boldsymbol{\psi}) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt + \int_0^T \int_\Omega (\boldsymbol{\psi} \cdot \nabla \mathbf{v}_m) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt \\ - \int_\Omega \mathbf{v}_m(0) \widetilde{\mathbf{v}}_j \varphi(0)\, d\mathbf{x} = \int_0^T \widetilde{\phi}_j f(\mathbf{v}_m, \alpha_{0m}) \varphi(t)\, dt, \quad \forall\, (\widetilde{\mathbf{v}}_j, \widetilde{\phi}_j) \in W(Q). \end{aligned} \tag{2.74}$$

Firstly, the integrals in the left hand side of (2.74) are examined. Using the weak estimates (2.60) leads to

$$\begin{aligned} \int_0^T \int_\Omega \mathbf{v}_m \cdot \widetilde{\mathbf{v}}_j \varphi'(t)\, d\mathbf{x}dt &\xrightarrow[m\to+\infty]{} \int_0^T \int_\Omega \mathbf{v} \cdot \widetilde{\mathbf{v}}_j \varphi'(t)\, d\mathbf{x}dt, \\ \int_0^T \int_\Omega \nabla \mathbf{v}_m : \nabla \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt &\xrightarrow[m\to+\infty]{} \int_0^T \int_\Omega \nabla \mathbf{v} : \nabla \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt, \\ \int_0^T \int_\Omega (\mathbf{v}_m \cdot \nabla \boldsymbol{\psi}) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt &\xrightarrow[m\to+\infty]{} \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \boldsymbol{\psi}) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt, \\ \int_0^T \int_\Omega (\boldsymbol{\psi} \cdot \nabla \mathbf{v}_m) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt &\xrightarrow[m\to+\infty]{} \int_0^T \int_\Omega (\boldsymbol{\psi} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt, \end{aligned}$$

for the linear terms. Further, since $\mathbf{v}_m$ converges to $\mathbf{v}$ in $L^2(0,T;\mathbf{V}(\Omega))$ weakly, and in $L^2(0,T;\mathbf{L}^2(\Omega))$ strongly, we can pass to the limit in the nonlinear term to obtain

$$\int_0^T \int_\Omega (\mathbf{v}_m \cdot \nabla \mathbf{v}_m) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt \xrightarrow[m\to+\infty]{} \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt. \tag{2.75}$$

The first and last terms in the left hand side of (2.74) are treated in the same manner.

Secondly, the integral in the right hand side of (2.74) is examined. Using Lemma 2.2, according to (2.59-a), we have $\alpha_{0m} \in L^\infty(0,T)$. Hence, for a subsequence of α_{0m} (still denoted by α_{0m}) :

$$\alpha_{0m} \rightharpoonup \alpha \text{ weakly}^* \text{ in } L^\infty(0,T). \tag{2.76}$$

Note that the convergence of $\mathbf{v}_m$ in $L^2([0,T]\times\Omega)$ implies its convergence in $L^1(0,T;\mathbf{L}^2(\Omega))$, i.e.

$$\|\mathbf{v}_m\| \longrightarrow \|\mathbf{v}\| \text{ in } L^1(0,T). \tag{2.77}$$

Due to Lemma 2.2, we have

$$|\alpha_{0_p} - \alpha_{0_q}| \leq C_b \|\mathbf{v}_p - \mathbf{v}_q\|, \quad \forall (\mathbf{v}_p, \alpha_{0p}), (\mathbf{v}_q, \alpha_{0_q}) \in W_m,$$

hence, α_{0_m} is then a Cauchy sequence in $L^1(0,T)$ and

$$\alpha_{0_m} \longrightarrow \phi \text{ in } L^1(0,T). \tag{2.78}$$

Moreover, according to (2.76) and using [14, Proposition II.1.26], we have $\phi = \alpha \in L^\infty(0,T)$. Furthermore, since α_{0_m} is bounded in $L^\infty(0,T)$, by (2.78) and [14, Corollaire II.1.24], we obtain $\alpha_{0_m} \longrightarrow \alpha$ in $L^p(0,T)$ for all $p \in]1, +\infty[$.

Now we can pass to the limit in the following terms :

$$\int_0^T \widetilde{\phi}_j \alpha_{0m}^2 \varphi(t) dt \quad \xrightarrow[m \to +\infty]{} \quad \int_0^T \widetilde{\phi}_j \alpha^2 \varphi(t) dt, \tag{2.79}$$

$$\int_0^T \widetilde{\phi}_j \alpha_{0m} \|\mathbf{v}_m\|^2 \varphi(t) dt \quad \xrightarrow[m \to +\infty]{} \quad \int_0^T \widetilde{\phi}_j \alpha \|\mathbf{v}\|^2 \varphi(t) dt, \tag{2.80}$$

and since $A_{\mathbf{z}_m} = b(\mathbf{w}, \boldsymbol{\psi}, \mathbf{z}_m) + b(\mathbf{z}_m, \boldsymbol{\psi}, \mathbf{w})$ with $\mathbf{z}_m = \mathbf{v}_m - \alpha_{0m}\mathbf{w}$, we have

$$\int_0^T \widetilde{\phi}_j \langle \mathbf{w}_l, \mathbf{z}_m \rangle \varphi(t) dt \quad \xrightarrow[m \to +\infty]{} \quad \int_0^T \widetilde{\phi}_j \langle \mathbf{w}, \mathbf{z} \rangle \varphi(t) dt, \tag{2.81}$$

$$\int_0^T \widetilde{\phi}_l A_{\mathbf{z}_m} \varphi(t) dt \quad \xrightarrow[m \to +\infty]{} \quad \int_0^T \widetilde{\phi}_l A_{\mathbf{z}} \varphi(t) dt, \tag{2.82}$$

where $A_{\mathbf{z}} = b(\mathbf{w}, \boldsymbol{\psi}, \mathbf{z}) + b(\mathbf{z}, \boldsymbol{\psi}, \mathbf{w})$. Finally,

$$\int_0^T \widetilde{\phi}_j f(\mathbf{v}_m, \alpha_{0m}) \varphi(t) dt \quad \xrightarrow[m \to +\infty]{} \quad \int_0^T \widetilde{\phi}_j f(\mathbf{v}, \alpha) \varphi(t) dt,$$

where $f(\mathbf{v}, \alpha) = A_{\mathbf{z}} + B_s\alpha + a_b\alpha^2 + b_b\alpha - 2\lambda_\nu \langle \mathbf{w}, \mathbf{z} \rangle - \big(\lambda_\nu \|\mathbf{w}\|^2 + K\|\mathbf{v}\|^2\big)\alpha$.

Consequently, passing to the limit in (2.74) leads to

$$\begin{aligned}
&- \int_0^T \int_\Omega \mathbf{v} \cdot \widetilde{\mathbf{v}}_j \varphi'(t)\, d\mathbf{x}dt + \nu \int_0^T \int_\Omega \nabla \mathbf{v} : \nabla \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt + \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt \\
&\quad + \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \boldsymbol{\psi}) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt + \int_0^T \int_\Omega (\boldsymbol{\psi} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\, d\mathbf{x}dt - \int_\Omega \mathbf{v}_0 \widetilde{\mathbf{v}}_j \varphi(0)\, d\mathbf{x} \\
&\qquad = \int_0^T \widetilde{\phi}_j \delta_{0_j} f(\mathbf{v}, \alpha)(t) \varphi(t)\, dt,
\end{aligned} \tag{2.83}$$

for all $\widetilde{\mathbf{v}}_j = \widetilde{\phi}_j \mathbf{w}_j,\ j \in \mathbb{N}$. By linearity, equation (2.83) holds true for all $\widetilde{\mathbf{v}}$ combination of finite $\widetilde{\mathbf{v}}_j$ and by density, for any element of $W(Q)$. $\square$

We now intend to prove the existence of the pressure.

4.4 Existence of the Pressure

First, we recall a result obtained in [33]

Lemma 4.8. *Let* $\mathbf{f} \in \mathcal{D}'(]0,T[;\mathbf{H}^{-1}(\Omega))$ *such that* $\langle \mathbf{f}, \widetilde{\mathbf{v}}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} = 0\ \forall \widetilde{\mathbf{v}} \in \mathbf{V}_0(\Omega)$. *Then, there exists* $q \in \mathcal{D}'(]0,T[;\mathbf{L}^2(\Omega))$ *such that* $\mathbf{f} = \nabla q$.

Lemma 4.8 is employed to prove the following result

Lemma 4.9. *There exists* $p \in \mathcal{D}'(]0,T[;L^2(\Omega))$ *such that* $(\mathbf{v},p)$ *satisfies (2.8-a) in the distribution sense.*

Proof. By choosing $\varphi \in \mathcal{D}(0,T)$ in (2.83), $\forall(\widetilde{\mathbf{v}},\widetilde{\alpha}) \in W(Q)$, we obtain

$$\begin{aligned}\int_0^T\int_\Omega \frac{\partial \mathbf{v}}{\partial t}\cdot\widetilde{\mathbf{v}}\,\varphi(t)\,d\mathbf{x}dt + \nu\int_0^T\int_\Omega \nabla\mathbf{v}:\nabla\widetilde{\mathbf{v}}\,\varphi(t)\,d\mathbf{x}dt + \int_0^T\int_\Omega(\mathbf{v}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\,\varphi(t)\,d\mathbf{x}dt\\ + \int_0^T\int_\Omega(\mathbf{v}\cdot\nabla\boldsymbol{\psi})\cdot\widetilde{\mathbf{v}}\,\varphi(t)\,d\mathbf{x}dt + \int_0^T\int_\Omega(\boldsymbol{\psi}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\,\varphi(t)\,d\mathbf{x}dt\\ = \int_0^T \widetilde{\alpha} f(\mathbf{v},\alpha)\,\varphi(t)dt.\end{aligned} \tag{2.84}$$

Further, taking $\widetilde{\alpha} = 0$ we have $\widetilde{\mathbf{v}} \in \mathbf{V}_0(\Omega)$, and from (2.84) we deduce

$$\begin{aligned}\int_\Omega \frac{\partial \mathbf{v}}{\partial t}\cdot\widetilde{\mathbf{v}}\,d\mathbf{x} + \nu\int_\Omega \nabla\mathbf{v}:\nabla\widetilde{\mathbf{v}}\,d\mathbf{x} + \int_\Omega(\mathbf{v}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\,d\mathbf{x} + \int_\Omega(\mathbf{v}\cdot\nabla\boldsymbol{\psi})\cdot\widetilde{\mathbf{v}}\,d\mathbf{x}\\ + \int_\Omega(\boldsymbol{\psi}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\,d\mathbf{x} = 0,\ \text{in } \mathcal{D}'(0,T).\end{aligned} \tag{2.85}$$

Then, letting

$$\mathbf{f} = \frac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\boldsymbol{\psi} + (\boldsymbol{\psi}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v},$$

and using (2.85), we obtain $\mathbf{f} \in \mathcal{D}'(]0,T[;\,\mathbf{H}^{-1}(\Omega))$ and $\langle \mathbf{f}, \widetilde{\mathbf{v}}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} = 0,\ \forall\widetilde{\mathbf{v}} \in \mathbf{V}_0(\Omega)$. Finally, using Lemma 4.8, there exists $p \in \mathcal{D}'(]0,T[;\,L^2(\Omega))$ such that $\mathbf{f} = -\nabla p$. $\square$

We now intend to retrieve the stabilized problem (2.8).

First, we prove that $(\mathbf{v},p)$ satisfies (2.8-f). Let us define the space

$$E(\Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \nabla\cdot\mathbf{u} \in L^2(\Omega)\},$$

and recall the following Lemma obtained in [33, Chap I, Theorem 1.2] :

Lemma 4.10. *Let Ω be an open bounded set of class $\mathcal{C}^2$. Then there exists a linear continuous operator $\gamma_n \in \mathcal{L}(E(\Omega), \mathbf{H}^{-1/2}(\Gamma))$ such that*

$$\gamma_n \mathbf{u} = \text{ the restriction of } \mathbf{u}\cdot\mathbf{n} \text{ to } \Gamma, \text{ for every } \mathbf{u} \in \mathcal{D}(\bar{\Omega}).$$

The following generalized Stokes formula is true for all $\mathbf{u} \in E(\Omega)$ and $\mathbf{w} \in \mathbf{H}^1(\Omega)$,

$$(\mathbf{u}, \nabla \mathbf{w}) + (\nabla \cdot \mathbf{u}, \mathbf{w}) = \langle \gamma_n \mathbf{u}, \gamma_0 \mathbf{w} \rangle, \tag{2.86}$$

where $\gamma_0 \in \mathcal{L}(\mathbf{H}^1(\Omega), \mathbf{L}^2(\Gamma))$ is the trace operator.

By writing (2.8-a) in the form

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot (-\nu \nabla \mathbf{v} + Ip) + (\mathbf{v}\cdot\nabla)\boldsymbol{\psi} + (\boldsymbol{\psi}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = 0 \ \text{ in } Q,$$

and using Lemma 4.10, we obtain

$$\begin{aligned}
&\int_\Omega \frac{\partial \mathbf{v}}{\partial t}\cdot \widetilde{\mathbf{v}}\, d\mathbf{x} + \int_\Omega (\nu\nabla \mathbf{v} - Ip) : \nabla \widetilde{\mathbf{v}}\, d\mathbf{x} + \langle(-\nu\nabla\mathbf{v} + Ip)\cdot\mathbf{n}, \widetilde{\mathbf{v}}\rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} \\
&\quad + \int_\Omega (\mathbf{v}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\, d\mathbf{x} + \int_\Omega (\mathbf{v}\cdot\nabla\boldsymbol{\psi})\cdot\widetilde{\mathbf{v}}\, d\mathbf{x} + \int_\Omega (\boldsymbol{\psi}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\, d\mathbf{x} = 0,
\end{aligned}$$

$\forall(\widetilde{\mathbf{v}}, \widetilde{\alpha}) \in W(Q)$. Letting $\widetilde{\mathbf{v}} = \widetilde{\alpha}\mathbf{w} \in W(Q)$ yields

$$\begin{aligned}
pI : \nabla\widetilde{\mathbf{v}} &= p\nabla\cdot\widetilde{\mathbf{v}} = 0, \\
\langle(-\nu\nabla\mathbf{v} + Ip)\cdot\mathbf{n}, \widetilde{\mathbf{v}}\rangle_{\mathbf{H}^{-\frac{1}{2}}(\Gamma), \mathbf{H}^{\frac{1}{2}}(\Gamma)} &= -\widetilde{\alpha}\int_{\Gamma_b} [\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}]\cdot\mathbf{g}\, d\zeta.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_\Omega \frac{\partial \mathbf{v}}{\partial t}\cdot\widetilde{\mathbf{v}}\, d\mathbf{x} + \nu\int_\Omega \nabla\mathbf{v} : \nabla\widetilde{\mathbf{v}}\, d\mathbf{x} + \int_\Omega (\mathbf{v}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\, d\mathbf{x} + \int_\Omega (\mathbf{v}\cdot\nabla\boldsymbol{\psi})\cdot\widetilde{\mathbf{v}}\, d\mathbf{x} \\
&\quad + \int_\Omega (\boldsymbol{\psi}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\, d\mathbf{x} = \widetilde{\alpha}\int_{\Gamma_b} [\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}]\cdot\mathbf{g}\, d\zeta.
\end{aligned} \tag{2.87}$$

By comparing (2.84) and (2.87), we retreive (2.8-f), namely

$$\int_{\Gamma_b} [\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}]\cdot\mathbf{g}\, d\zeta = f(\mathbf{v}, \alpha).$$

Finally, it remains to verify that the initial condition (2.8-e) belongs to $\mathbf{H}(\Omega)$. In this purpose, we multiply (2.8-a) by $\widetilde{\mathbf{v}}\varphi$, with $\varphi(T) = 0$, and integrate with respect to time and space

$$
\begin{aligned}
&- \int_0^T \int_\Omega \mathbf{v}\cdot\widetilde{\mathbf{v}}\,\varphi'(t)\,d\mathbf{x}dt + \nu\int_0^T\int_\Omega \nabla\mathbf{v} : \nabla\widetilde{\mathbf{v}}\,\varphi(t)\,d\mathbf{x}dt + \int_0^T\int_\Omega (\mathbf{v}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\,\varphi(t)\,d\mathbf{x}dt \\
&\quad + \int_0^T\int_\Omega (\mathbf{v}\cdot\nabla\boldsymbol{\psi})\cdot\widetilde{\mathbf{v}}\,\varphi(t)\,d\mathbf{x}dt + \int_0^T\int_\Omega (\boldsymbol{\psi}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\,\varphi(t)\,d\mathbf{x}dt - \int_\Omega \mathbf{v}(0)\widetilde{\mathbf{v}}\,\varphi(0)\,d\mathbf{x} \\
&\qquad = \int_0^T \widetilde{\alpha} f(\mathbf{v},\alpha)(t)\,\varphi(t)dt.
\end{aligned}
\tag{2.88}
$$

By comparing (2.83) and (2.88), we obtain $\int_\Omega(\mathbf{v}(0)-\mathbf{v}_0)\cdot\widetilde{\mathbf{v}}\,\varphi(0)\,d\mathbf{x} = 0$, and choosing φ such that $\varphi(0) = 1$, yields

$$
\int_\Omega (\mathbf{v}(0)-\mathbf{v}_0)\cdot\widetilde{\mathbf{v}}\,d\mathbf{x} = 0 \quad \forall(\widetilde{\mathbf{v}},\widetilde{\alpha}) \in W(Q).
$$

Hence, as in chapter 1, $\mathbf{v}(0) = \mathbf{v}_0$ in $\mathcal{E}(Q)$ which is defined in (1.75).

5 Concluding remarks

In this paper, the exponential stabilization of the two and three-dimensional Navier-Stokes equations in a bounded domain is studied around a given *non-stationary state* flow, using a boundary feedback control. In order to determine a feedback law, an extended system coupling the Navier-Stokes equations with an equation satisfied by the control on the domain boundary is considered. We first assume that on Σ_b (a part of the domain boundary), the trace of the fluid velocity is proportional to a given velocity profile g. The proportionality coefficient α measures the velocity flux at the interface. It is an unknown of the problem and is written in feedback form. By using the Galerkin method, α is determined such that the Dirichlet boundary control $\mathbf{v}_b = \alpha\mathrm{g}$ is satisfied on Σ_b, and the stabilizing boundary control is built. We show that the nonlinear feedback control provides global exponential stabilization of the non-stationary state belonging in the set *admissible target velocities*. This feedback control was shown to guarantee global stability in the L^2-norm.

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Chapitre 3

Feedback stabilization of the Navier-Stokes system with mixed boundary conditions

Abstract

This paper presents a boundary feedback control for the two and three-dimensional Navier-Stokes equations in a bounded domain Ω with mixed boundary conditions around a given steady-state flow. In order to determine a feedback control law, we consider an extended system coupling the equations governing the perturbation with an equation satisfied by the control on the domain boundary. By using the Faedo-Galerkin method and a priori estimation techniques, a stabilizing boundary control is built. This control law ensures a decrease of the energy of the controlled nonlinear discrete system. A compactness result then allows us to pass to the limit in the system satisfied by the approximated solutions.

Keywords : Navier-Stokes system, feedback control, boundary stabilization, Galerkin method.

1 Introduction

Let Ω be a bounded and connected domain in $\mathbb{R}^d,\ d = 2, 3$, with a boundary Γ of class C^2, and made up of three connected components Γ_l, Γ_e and Γ_s with $\Gamma = \Gamma_l \cup \Gamma_e \cup \Gamma_s$. Such a boundary decomposition is schematized in Figure 3.1. In particular, the boundary Γ_e is the part of Γ, where a Dirichlet boundary control in feedback form has to be determined. The usual function spaces $L^2(\Omega),\ H^1(\Omega),\ H^1_0(\Omega)$ are used and we let $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{H}^1(\Omega) = (H^1(\Omega))^d,\ \mathbf{H}^1_0(\Omega) = (H^1_0(\Omega))^d$. Negative ordered Sobolev spaces $\mathbf{H}^{-1}(\Omega)$ are defined as the dual space, i.e., $\mathbf{H}^{-1}(\Omega) = \{\mathbf{H}^1_0(\Omega)\}'$. We denote by $\langle \cdot \mid \cdot \rangle$ and $\| \cdot \| = \| \cdot \|_{\mathbf{L}^2(\Omega)}$, the scalar product and norm in $\mathbf{L}^2(\Omega)$, respectively. Further, if $\mathbf{u} \in \mathbf{L}^2(\Omega)$ is such that $\nabla \cdot \mathbf{u} \in L^2(\Omega)$, we denote the normal trace of $\mathbf{u}$ in $\mathbf{H}^{-\frac{1}{2}}(\Gamma)$ by $\mathbf{u} \cdot \mathbf{n}$, where $\mathbf{n}$ denotes the unit outer normal vector to Γ.

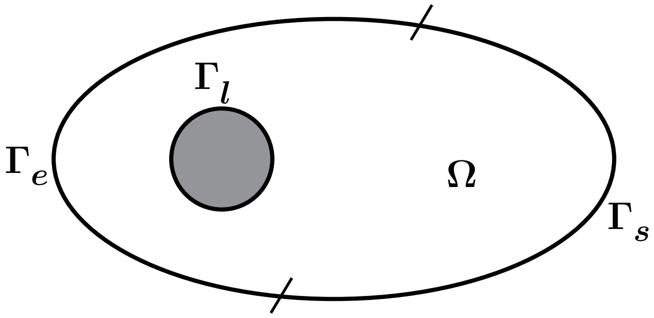

FIGURE 3.1 – Description of the domain Ω and of the three connected components Γ_e, Γ_l and Γ_s.

In order to define the stabilization problem, we consider a velocity-pressure pair $(\mathbf{v}_s, q_s)$ solution to the stationary Navier-Stokes equations

$$\begin{cases} -\nu\Delta\mathbf{v}_s + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s + \nabla q_s = \mathbf{f}_s & \text{in } \Omega, \\ \nabla \cdot \mathbf{v}_s = 0 & \text{in } \Omega, \\ \mathbf{v}_s = 0 & \text{on } \Gamma_l, \\ \mathbf{v}_s = \boldsymbol{\psi}_e & \text{on } \Gamma_e, \\ \nu\nabla\mathbf{v}_s \cdot \mathbf{n} - q_s\mathbf{n} = \boldsymbol{\psi}_s & \text{on } \Gamma_s, \end{cases} \tag{3.1}$$

where $\nu > 0$ is the viscosity coefficient, $\mathbf{f}_s$ represents the body forces acting on the fluid, $\boldsymbol{\psi}_e$ is the Dirichlet boundary condition on Γ_e and $\boldsymbol{\psi}_s$ is the Neumann boundary condition on Γ_s. Further, we assume that couple $(\mathbf{v}_s, q_s)$ belongs to $\mathbf{H}^1(\Omega) \times L^2_0(\Omega)$, where $L^2_0(\Omega)$ define the pressure space with zero mean value :

$$L^2_0(\Omega) = \left\{ p \in L^2(\Omega),\ \int_\Omega p(\mathbf{x})\, d\mathbf{x} = 0 \right\}.$$

Let us first define the set *admissible target velocities* $\mathcal{U}_{ad}$. The solution $\mathbf{v}_s$ of (3.1) is said to be in the set admissible target velocities $\mathcal{U}_{ad}$ if

$$\|\nabla\mathbf{v}_s\| < \frac{\nu}{M_p}, \tag{3.2}$$

where M_p is a positive constant defined later in (3.37).

For $T > 0$ fixed, we take $Q = [0,T) \times \Omega$, $\Sigma_l = [0,T) \times \Gamma_l$, $\Sigma_e = [0,T) \times \Gamma_e$ and $\Sigma_s = [0,T) \times \Gamma_s$, and we consider the velocity-pressure pair $(\mathbf{u}, q)$ solution to the non stationary

Navier-Stokes equations

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla q = \mathbf{f}_s & \text{in } Q, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } Q, \\ \mathbf{u}(t,\mathbf{x}) = 0 & \text{on } \Sigma_l, \\ \mathbf{u}(t,\mathbf{x}) = \mathbf{u}_e(t,\mathbf{x}) + \boldsymbol{\psi}_e(\mathbf{x}) & \text{on } \Sigma_e, \\ \nu\nabla\mathbf{u}(t,\mathbf{x})\cdot\mathbf{n} - q(t,\mathbf{x})\mathbf{n} = \mathbf{u}_s(t,\mathbf{x}) + \boldsymbol{\psi}_s(\mathbf{x}) & \text{on } \Sigma_s, \\ \mathbf{u}(t=0,\mathbf{x}) = \mathbf{v}_s(\mathbf{x}) + \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{3.3}$$

where $\mathbf{u}_s$ is a given Neumann boundary condition on Σ_s which is defined later, $\mathbf{u}_e$ is the control input which is built later and $\mathbf{v}_0(\mathbf{x})$ is the initial perturbation and the initial condition of $\mathbf{v}$ in (3.4), it belongs to an appropriate functional space which will be defined later.

Problems of type (3.3) have already been investigated in the literature. For example, in [14, 15, 24], energy estimates in velocity-pressure are established and a proof of existence of solutions is obtained, where the Neumann boundary condition on Σ_s is chosen appropriately. In [14, 15] the Neumann condition is derived from a weak formulation, while in [24], it is treated by pseudo-differential methods. However, the studies in [14, 15, 24] are only concerned with the existence of velocity-pressure solutions and not with a stabilization problem by means of a boundary feedback control, which is the subject of the present paper.

By substituting $\mathbf{u} = \mathbf{v} + \mathbf{v}_s$ and $q = p + q_s$ in (3.3), the resulting system is obtained for the velocity-pressure pair $(\mathbf{v}, p)$

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}_s + (\mathbf{v}_s\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0 & \text{in } Q, \\ \nabla\cdot\ \mathbf{v} = 0 & \text{in } Q, \\ \mathbf{v} = 0 & \text{on } \Sigma_l, \\ \mathbf{v}(t,\mathbf{x}) = \mathbf{u}_e(t,\mathbf{x}) & \text{on } \Sigma_e, \\ \nu\nabla\mathbf{v}\cdot\mathbf{n} - p\mathbf{n} = \mathbf{u}_s(t,\mathbf{x}) & \text{on } \Sigma_s, \\ \mathbf{v}(t=0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{3.4}$$

where $\mathbf{u}_s$ is a given Neumann boundary condition on Σ_s defined later.

Our goal is the following : for a prescribed rate of decrease $\sigma > 0$, we need to find a feedback control $\mathbf{u}_e$ on Σ_e such that the velocity $\mathbf{v}$ in (3.4) satisfies the exponential decay

$$\|\mathbf{v}(t)\| \le \|\mathbf{v}_0\| e^{-\sigma(t)}, \quad t \in (0,\infty). \tag{3.5}$$

Note that $\sigma(t)$ is usually written as $\sigma_0 t$ in previous studies, where σ_0 is positive constant. The control $\mathbf{u}_e(t)$ is called a feedback if there exists a mapping $\mathcal{M} : \mathbf{X}(\Omega) \to \mathbf{U}(\Gamma_e)$ such that

$$\mathbf{u}_e(t) = \mathcal{M}(\mathbf{v}(t)), \quad t \in (0, \infty), \tag{3.6}$$

where the spaces $\mathbf{X}(\Omega)$ and $\mathbf{U}(\Gamma_e)$ are defined in the sequel.

The theoretical setting of the stabilization procedure, for the non stationary incompressible Navier-Stokes equations using a feedback control, has been studied by a number of authors, e.g. A.V. Fursikov [17, 18], V. Barbu et al. [6, 10, 11, 12, 13], J.-P. Raymond et al. [28, 29, 30] and M. Badra et al. [2, 4, 5]. In these papers, the authors consider the Dirichlet condition only and system (3.4) is written in the form

$$\mathbf{y}'(t) = A\mathbf{y}(t) + B\mathbf{u}(t) + \kappa F(\mathbf{y}(t), \mathbf{u}(t)), \quad \mathbf{y}(t=0) = \mathbf{y}_0. \tag{3.7}$$

where $\mathbf{y}(t)$ is the new state variable, $\mathbf{u}(t)$ the new control variable, A is a linear operator which is the infinitesimal generator of an analytic semigroup, B is a linear operator, F is a nonlinear mapping and $\kappa = 0$ or 1. Further, the linear feedback law $\mathcal{M}$ is first determined by solving a linear control problem for the linearized system of equations, i.e. $\kappa = 0$ in (3.7), and then this linear feedback is used in order to stabilize the original non linear system i.e. $\kappa = 1$ in (3.7).

By employing the extension operator, A.V. Fursikov [17, 18] addressed the stabilization of the $2D$ and $3D$ Navier-Stokes equations. In [4, 5, 9, 10, 11, 29, 30], the feedback control laws are determined by solving a Riccati equation in a space of infinite dimension. In such a case, an optimal control problem has to be solved, involving the minimization of an objective functional. In practice, the control is calculated through approximation via the solution of an algebraic Riccati equation, which may be computationally expensive. The use of finite-dimensional controllers may be more appropriate to stabilize the Navier-Stokes equations. Such an approach is performed in [12], in the case of an internal control, and in [2, 9, 10, 11, 28], in the case of a boundary control. In [2, 12, 28], the authors search for a boundary control $\mathbf{u}_e$ of finite dimension of the form

$$\mathbf{u}_e = \sum_{j=1}^{N} u_j(t)\boldsymbol{\psi}_j(\mathbf{x}), \qquad t \geq 0,\ x \in \Gamma,$$

where $(\psi_j)_{j=1,2,3,\ldots,N}$ is a finite-dimensional basis obtained from the eigenfunctions of some operator and $\overline{\mathbf{u}} = (u_1, u_2, u_3, \ldots, u_N)$ is a control function expressed with a feedback formulation. In [28] and [2], where $d = 2$, and $d = 3$, respectively, the feedback control is obtained from the solution of a finite-dimensional Riccati equation while a stochastic-based stabilization technique is employed in [8], in the case of an internal control, which avoids the difficult computation problems related to infinite-dimensional Riccati equations. The procedure employed in [6] for a boundary control ressembles the form of stabilizing noise controllers designed in [8].

A linear feedback law is first determined by solving a linear control problem in all the papers cited above, and this linear feedback is then used in order to stabilize the original non linear system. Such a procedure leads to choose the initial velocity small enough and it usually requires to search for the control $\mathbf{u}_e$ and the initial condition in sufficiently regular spaces. This is why another approach is proposed in [26], where an extended system is considered with an additional equation satisfied by the control on the domain boundary, and the boundary feedback control is constructed via a Galerkin method. In [26], the system is not written in the form (3.7) and the control law is not determined by solving a linear problem. Accordingly, the authors obtain a stability result for an arbitrary initial data in an appropriate space and for prescribed rate of decrease $\sigma > 0$, which depends on the viscosity ν.

In this paper, the approach of [26], using an extended system is followed, but instead of considering Dirichlet boundary conditions on the whole domain boundary, mixed Dirichlet-Neumann boundary conditions are employed instead. The Dirichlet and Neumann conditions are imposed on $\Gamma_l \cup \Gamma_e$ and Γ_s, respectively. However, as in [26], the control is imposed only on a part of the Dirichlet boundary, namely Γ_e. Such a mixed Dirichlet-Neumann feedback stabilization problem is new, to our knowledge, and the problem seems to have been considered only numerically in [1, 3].

The boundary control $\mathbf{u}_e$ in (3.4) is rewritten on the form $\mathbf{u}_e = \alpha(t)\mathbf{g}(\mathbf{x})$ on Σ_e, where α is a priori unknown, $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$ is assumed to verify $\mathbf{g} = 0$ on $\Gamma_l \cup \Gamma_s$, $\mathbf{g} \cdot \mathbf{n} \neq 0$ on Γ_e and $\int_{\Gamma_e} \mathbf{g} \cdot \mathbf{n} = 0$. In order to stabilize (3.4), with $\mathbf{u}_e = \alpha(t)\mathbf{g}(\mathbf{x})$ on Σ_e, by employing energy a priori estimation technics, the quantity $\alpha(t)$ is found to satisfy the relation

$$\int_{\Gamma_e} [\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}] \cdot \mathbf{g} \, d\zeta = \mathcal{F}(\mathbf{v}, \alpha). \tag{3.8}$$

where $\mathcal{F}$ is a polynomial in α of degree 2 to be defined later. The quantity $\alpha(t)$ depends nonlinearly on $\mathbf{v}$ in (3.8), and hence $\alpha(t)$ satisfies a nonlinear feedback law of the form (3.6), and the mapping $\mathcal{M}$ is nonlinear. System (3.4) is then extended by adding (3.8), and the extended system, namely (3.4) and (3.8), with $\mathbf{u}_e = \alpha(t)\mathbf{g}(\mathbf{x})$ on Σ_e, is the stabilization problem considered in this paper, i.e.

$$\begin{cases} (a) & \dfrac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}_s + (\mathbf{v}_s\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0 & \text{in } Q, \\ (b) & \nabla\cdot\ \mathbf{v} = 0 & \text{in } Q, \\ (c) & \mathbf{v} = 0 & \text{on } \Sigma_l, \\ (d) & \mathbf{v} = \alpha(t)\mathbf{g}(\mathbf{x}) & \text{on } \Sigma_e, \\ (e) & \nu\nabla\mathbf{v}\cdot\mathbf{n} - p\mathbf{n} = \mathbf{u}_s(t,\mathbf{x}) & \text{on } \Sigma_s, \\ (f) & \displaystyle\int_{\Gamma_e} [\nu\nabla\mathbf{v}\cdot\mathbf{n} - p\mathbf{n}]\cdot\mathbf{g}\, d\zeta = \mathcal{F}(\mathbf{v},\alpha), & \\ (g) & \mathbf{v}(t=0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{3.9}$$

In order to determined $\alpha(t)$, leading to the determination of the boundary control $\mathbf{u}_e$, system (3.9) is solved via a Galerkin procedure which consists on building a sequence of approximated solutions using an adequate Galerkin basis.

The paper is organized as follows. In section 2, the notations and mathematical preliminaries are given. In section 3, thanks to technics developed in [23] (which are not related specifically to a stabilization problem), the existence of at least one weak solution of the stabilization problem (3.9) is established by applying the Galerkin method.

2 Notation and Preliminaries

2.1 Function Spaces

Several spaces of free divergence functions are now introduced :

$$\begin{aligned} \mathcal{V}(\Omega) &= \{\mathbf{u}\in\mathcal{D}(\Omega),\ \nabla\cdot\mathbf{u} = 0\}, && (3.10)\\ \mathbf{V}_0(\Omega) &= \text{the closure of } \mathcal{V}(\Omega) \text{ in } \mathbf{H}^1_0(\Omega), && (3.11)\\ \mathbf{V}(\Omega) &= \left\{\mathbf{u}\in\mathbf{H}^1(\Omega)\ :\nabla\cdot\mathbf{u} = 0,\ \ \mathbf{u} = 0 \text{ on } \Gamma_l\right\}, && (3.12)\\ \mathbf{Z}(\Omega) &= \left\{\mathbf{u}\in\mathbf{H}^1(\Omega)\ :\nabla\cdot\mathbf{u} = 0,\ \ \mathbf{u} = 0 \text{ on } \Gamma_l\cup\Gamma_e\right\}, && (3.13)\\ \mathbf{H}(\Omega) &= \left\{\mathbf{u}\in\mathbf{L}^2(\Omega)\ :\nabla\cdot\mathbf{u} = 0,\ \ \mathbf{u} = 0 \text{ on } \Gamma_l\right\}. && (3.14) \end{aligned}$$

Remark 2.1. *Since $\mathbf{V}(\Omega)$ and $\mathbf{Z}(\Omega)$ are closed subspaces of $\mathbf{H}^1(\Omega)$, we have by definition,*

$$\|\cdot\|_{\mathbf{V}(\Omega)} = \|\cdot\|_{\mathbf{Z}(\Omega)} = \|\cdot\|_{\mathbf{H}^1(\Omega)}.$$

Remark 2.2. *Since $\mathbf{Z}(\Omega)$ is a closed subspace of $\mathbf{H}^1(\Omega)$, it follows that $\mathbf{Z}(\Omega)$ is a separable Hilbert space and thus $\mathbf{Z}(\Omega)$ admits a countable orthonormal basis $(\mathbf{z}_n)_{n\in\mathbb{N}}$ which will be used in the sequel.*

Definition 2.3. *Let $V^{\frac{1}{2}}(\Gamma_e)$ be the space of functions whose extension by zero over Γ belong to $\mathbf{H}^{\frac{1}{2}}(\Gamma)$. Further, we define*

$$W(Q) = \{(\mathbf{v}, \alpha) \in \mathbf{V}(\Omega) \times \mathbb{R},\ s.t.\ \mathbf{v} = \alpha \mathbf{g} \text{ on } \Gamma_e\}, \tag{3.15}$$

where g *satisfies*

$$\mathbf{g} \in V^{\frac{1}{2}}(\Gamma_e), \tag{3.16}$$

$$\mathbf{g} \cdot \mathbf{n} \neq 0 \text{ on } \Gamma_e, \tag{3.17}$$

$$\int_{\Gamma_e} \mathbf{g} \cdot \mathbf{n}\, d\zeta = 0. \tag{3.18}$$

Remark 2.4. *The solution of (3.9) is searched in $W(Q)$, defined in (3.15).*

The following lemma [23], is used in the sequel.

Lemma 2.5. *There exists a constant $C_e > 0$ such that, for all $(\mathbf{v}, \alpha) \in W(Q)$, with* g *satisfying (3.52) and (3.17), we have*

$$|\alpha| \leq C_e \|\mathbf{v}\|. \tag{3.19}$$

Remark 2.6. *For all $(\mathbf{v}, \alpha) \in W(Q)$, inequality (3.19) of Lemma 2.5 holds with $(\widehat{\mathbf{v}}, \widehat{\alpha})$, where $\widehat{\mathbf{v}}$ and $\widehat{\alpha}$ denote the Fourier transforms of* v *and α, respectively.*

2.2 Linear Forms

In order to define a weak form of the stabilization problem, we introduce the continuous bilinear form

$$a(\mathbf{v}_1, \mathbf{v}_2) = \int_\Omega \nabla \mathbf{v}_1 : \nabla \mathbf{v}_2\, d\mathbf{x}, \quad \forall \mathbf{v}_j \in \mathbf{H}^1(\Omega),\ j = 1, 2,$$

and trilinear form

$$b(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \int_\Omega (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2 \cdot \mathbf{v}_3\, d\mathbf{x}, \quad \forall \mathbf{v}_j \in \mathbf{H}^1(\Omega),\ j = 1, 2.$$

Thanks to Hölder inequality, the functional b satisfies

$$|b(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)| \quad \leq \quad \|\mathbf{v}_1\|_{\mathbf{L}^3(\Omega)} \|\nabla \mathbf{v}_2\| \, \|\mathbf{v}_3\|_{\mathbf{L}^6(\Omega)}, \quad \forall \mathbf{v}_j \in \mathbf{H}^1(\Omega),\ j = 1, 2, 3.$$

Further, using the generalized Sobolev's inequality, leads to

$$\|\mathbf{v}_1\|_{\mathbf{L}^3(\Omega)} \leq C_1 \|\mathbf{v}_1\|^{\frac{1}{2}} \|\nabla \mathbf{v}_1\|^{\frac{1}{2}} \quad \text{and} \quad \|\mathbf{v}_3\|_{\mathbf{L}^6(\Omega)} \leq C_2 \|\nabla \mathbf{v}_3\|, \quad \text{for } d = 2, 3,$$

where C_i, $i = 1, 2$ is a positive constant, and for $C_s = C_1 C_2$, we have

$$|b(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)| \leq C_s \|\mathbf{v}_1\|^{\frac{1}{2}} \|\nabla \mathbf{v}_1\|^{\frac{1}{2}} \|\nabla \mathbf{v}_2\| \|\nabla \mathbf{v}_3\|. \tag{3.20}$$

3 Control building

In the first step a hilbertian basis for space $W(Q)$ defined in (3.15) is built.

3.1 A Galerkin basis for space $W(Q)$

We assume that the boundary Γ_e is composed of two connected components such that $\Gamma_e = \Gamma_0 \cup \Gamma_1$. Let g_0 such that $\mathrm{g}_0 \in V^{\frac{1}{2}}(\Gamma_0)$ and $\int_{\Gamma_0} \mathrm{g}_0 \cdot \mathbf{n} \neq 0$, we consider this problem

$$\begin{cases} (a) & -\Delta \mathbf{w} + \nabla q = 0, \quad \nabla \cdot \mathbf{w} = 0 & \text{in } \Omega, \\ (b) & \mathbf{w} = 0 & \text{on } \Gamma_l \cup \Gamma_s, \\ (c) & \mathbf{w} = \beta \mathrm{g}_0 & \text{on } \Gamma_0, \\ (d) & \mathbf{w} = \mathrm{g}_1 & \text{on } \Gamma_1, \end{cases} \tag{3.21}$$

where g_1 is such that $\mathrm{g}_1 \in V^{\frac{1}{2}}(\Gamma_1)$ with $\mathrm{g}_1 \cdot \mathbf{n} \neq 0$ on Γ_1 and $\beta = -\dfrac{\int_{\Gamma_1} \mathrm{g}_1 \cdot \mathbf{n}\, d\zeta}{\int_{\Gamma_0} \mathrm{g}_0 \cdot \mathbf{n}\, d\zeta}$. Further, let

$$\mathrm{g} = \begin{cases} \beta \mathrm{g}_0 & \text{on } \Gamma_0, \\ \mathrm{g}_1 & \text{on } \Gamma_1, \end{cases} \tag{3.22}$$

and hence, by construction, g satisfies (3.52)-(3.18). Since $\mathrm{w} = \mathrm{g}$ on $\Gamma_e = \Gamma_0 \cup \Gamma_1$, system (3.21) admits a unique solution (w, q) belonging to $\mathbf{H}^1(\Omega) \times L^2_0(\Omega)$ (see [14, Proposition III.4.1]). Moreover, for all $\mathbf{z} \in \mathbf{Z}(\Omega)$ defined in (3.13) and for all $\alpha \in \mathbb{R}$, we have $\mathbf{v} = \mathbf{z} + \alpha \mathrm{w} \in W(\Omega)$, where w satisfies (3.21). Indeed, we have $\mathbf{z}, \mathrm{w} \in \mathbf{V}(\Omega)$ and since $\mathbf{z} = 0$ on Γ_e, we obtain $\mathbf{v} = \alpha \mathrm{g}$ on Γ_e. Due to Remark 2.2, $\mathbf{Z}(\Omega)$ admits a countable orthonormal basis $(\mathbf{z}_n)_{n\in\mathbb{N}}$, the sequence $\mathrm{w},\ \mathbf{z}_1,\ \mathbf{z}_2,\ \mathbf{z}_3, \cdots$, is then linearly independent. Consequently, we search for the solution v of (3.9) in

$$W(Q) = \text{span}(\mathrm{w}) \oplus \text{span}(\mathbf{z}_n)_{\{n\in\mathbb{N}^*\}}, \tag{3.23}$$

and v can be expressed as : $\mathbf{v} = \alpha \mathrm{w} + \mathbf{z}$, with $\mathbf{z} = \sum_{i=1}^{\infty} \theta_i \mathbf{z}_i$.

3.2 The control building

Multiplying (3.9-a) by $\mathbf{v} = \alpha\mathbf{w} + \mathbf{z} \in W(Q)$ and integrating by parts over Ω, using (3.9-e) and (3.9-f) leads to

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\|\nabla\mathbf{v}\|^2 + b(\mathbf{v},\mathbf{v},\mathbf{v}) + b(\mathbf{v}_s,\mathbf{v},\mathbf{v}) + b(\mathbf{v},\mathbf{v}_s,\mathbf{v}) = \int_{\Gamma_s} \mathbf{u}_s \cdot \mathbf{z}\, d\zeta + \alpha\mathcal{F}(\mathbf{v},\alpha). \tag{3.24}$$

We now define the function $\mathbf{u}_s$ and build the control law $\mathcal{F}(\mathbf{v},\alpha)$ by rewriting the terms in the left hand side of (3.24) :

$$\|\mathbf{v}\|^2 = \alpha^2\|\mathbf{w}\|^2 + 2\,\alpha\langle\mathbf{w},\mathbf{z}\rangle + \|\mathbf{z}\|^2, \tag{3.25}$$

$$\|\nabla\mathbf{v}\|^2 = \alpha^2\|\nabla\mathbf{w}\|^2 + 2\,\alpha\langle\nabla\mathbf{w},\nabla\mathbf{z}\rangle + \|\nabla\mathbf{z}\|^2. \tag{3.26}$$

Integrating by parts the following trilinear forms, yields

$$b(\mathbf{v}_s,\mathbf{v},\mathbf{v}) = \frac{1}{2}\int_{\Gamma_s} |\mathbf{z}^2|\,\mathbf{v}_s\cdot\mathbf{n}\,d\zeta + \frac{\alpha^2}{2}\int_{\Gamma_e} |\mathbf{g}|^2\,\mathbf{v}_s\cdot\mathbf{n}\,d\zeta, \tag{3.27}$$

$$b(\mathbf{v},\mathbf{v},\mathbf{v}) = \frac{1}{2}\int_{\Gamma_s} |\mathbf{z}^2|\,\mathbf{z}\cdot\mathbf{n}\,d\zeta + \frac{\alpha^3}{2}\int_{\Gamma_e} |\mathbf{g}|^2\,\mathbf{g}\cdot\mathbf{n}\,d\zeta. \tag{3.28}$$

In order to define the Neumann boundary condition $\mathbf{u}_s$, we recall that for all $x \in \mathbb{R}$, we have

$$x = x^+ - x^-, \tag{3.29}$$

where $x^+ = \max(x,0)$ and $x^- = -\min(x,0)$. From (3.29), we have

$$\mathbf{v}_s\cdot\mathbf{n} = (\mathbf{v}_s\cdot\mathbf{n})^+ - (\mathbf{v}_s\cdot\mathbf{n})^-, \tag{3.30}$$

$$\mathbf{z}\cdot\mathbf{n} = (\mathbf{z}\cdot\mathbf{n})^+ - (\mathbf{z}\cdot\mathbf{n})^-, \tag{3.31}$$

and we define the function $\mathbf{u}_s$ on Γ_s by taking (see [14, Page 247])

$$\mathbf{u}_s = -\frac{1}{2}\,\mathbf{z}\,(\mathbf{v}_s\cdot\mathbf{n})^- - \frac{1}{2}\,\mathbf{z}\,(\mathbf{z}\cdot\mathbf{n})^-. \tag{3.32}$$

By substituting (3.27)-(3.28) and (3.32) in (3.24) and by using (3.30)-(3.31), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\|\nabla\mathbf{v}\|^2 + \frac{1}{2}\int_{\Gamma_s} |\mathbf{z}^2|(\mathbf{v}_s\cdot\mathbf{n})^+ d\zeta + \frac{1}{2}\int_{\Gamma_s} |\mathbf{z}^2|(\mathbf{z}\cdot\mathbf{n})^+ d\zeta + b(\mathbf{v},\mathbf{v}_s,\mathbf{v}) + a_e\alpha^3 + b_e\alpha^2 = \alpha\mathcal{F}(\mathbf{v},\alpha), \tag{3.33}$$

where

$$a_e = \frac{1}{2}\int_{\Gamma_e} |\mathbf{g}|^2(\mathbf{g}\cdot\mathbf{n})\, d\zeta, \quad b_e = \frac{1}{2}\int_{\Gamma_e} |\mathbf{g}|^2(\mathbf{v}_s\cdot\mathbf{n})\, d\zeta.$$

We now define the control law $\mathcal{F}$ as

$$\mathcal{F}(\mathbf{v},\alpha) = a_e\alpha^2 + b_e\alpha - \lambda_\nu\left(\alpha\|\mathbf{w}\|^2 + 2\langle\mathbf{w},\mathbf{z}\rangle\right) + 2\,\beta_\nu\langle\nabla\mathbf{w},\nabla\mathbf{z}\rangle - K\alpha\|\mathbf{v}\|^2, \tag{3.34}$$

where the positive constants λ_ν and β_ν will be defined later. Note that in (3.34), the terms $\alpha\|\mathbf{w}\|^2 + 2\langle\mathbf{w},\mathbf{z}\rangle$ and $\langle\nabla\mathbf{w},\nabla\mathbf{z}\rangle$ are derived from (3.25) and (3.26), respectively ; while the term $-K\alpha\|\mathbf{v}\|^2$ is introduced in order to limit the size of the control, for an appropriate choice of $K>0$.

4 Stability Result

We first establish the a priori estimates for the extended stabilization system (3.9).

4.1 A priori estimates

Multiplying (3.34) by α, substituting in (3.33) and using (3.25) leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\|\nabla\mathbf{v}\|^2 + \frac{1}{2}\int_{\Gamma_s}|\mathbf{z}^2|(\mathbf{v}_s\cdot\mathbf{n})^+ + \frac{1}{2}\int_{\Gamma_s}|\mathbf{z}^2|(\mathbf{z}\cdot\mathbf{n})^+ \\ + b(\mathbf{v},\mathbf{v}_s,\mathbf{v}) = -\lambda_\nu\left(\|\mathbf{v}\|^2 - \|\mathbf{z}\|^2\right) + 2\,\alpha\beta_\nu\langle\nabla\mathbf{w},\nabla\mathbf{z}\rangle - K\|\mathbf{v}\|^2\alpha^2.\end{aligned} \tag{3.35}$$

We obtain from (3.20) $|b(\mathbf{v},\mathbf{v}_s,\mathbf{v})| \le C_s\|\mathbf{v}\|^{\frac{1}{2}}\|\nabla\mathbf{v}\|^{\frac{1}{2}}\|\nabla\mathbf{v}_s\|\|\nabla\mathbf{v}\|$, and since $\mathbf{v}=0$ on Γ_l which is a part of the domain boundary, from the Poincaré inequality, there exists a constant C_p such that $\|\mathbf{v}\| \le C_p\|\nabla\mathbf{v}\|$, and hence

$$|b(\mathbf{v},\mathbf{v}_s,\mathbf{v})| \le M_p\|\nabla\mathbf{v}_s\|\|\nabla\mathbf{v}\|^2, \tag{3.36}$$

where

$$M_p = C_p^{\frac{1}{2}}C_s. \tag{3.37}$$

By taking $\beta_\nu = \nu - M_p\|\nabla\mathbf{v}_s\|$ which is assumed positive, using (3.36) in (3.35) and due to (3.26), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \beta_\nu\alpha^2\|\nabla\mathbf{w}\|^2 + \beta_\nu\|\nabla\mathbf{z}\|^2 + \frac{1}{2}\int_{\Gamma_s}|\mathbf{z}^2|(\mathbf{v}_s\cdot\mathbf{n})^+ + \frac{1}{2}\int_{\Gamma_s}|\mathbf{z}^2|(\mathbf{z}\cdot\mathbf{n})^+ \\ \le\; -\lambda_\nu\left(\|\mathbf{v}\|^2 - \|\mathbf{z}\|^2\right) - K\|\mathbf{v}\|^2\alpha^2.\end{aligned} \tag{3.38}$$

Moreover, since $\mathbf{z} = 0$ on $\Gamma_l \cup \Gamma_e$, we obtain $\|\mathbf{z}\| \leq C_p \|\nabla \mathbf{z}\|$ and taking $\lambda_\nu = \frac{\beta_\nu}{C_p^2}$ in (3.38) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \beta_\nu \alpha^2 \|\nabla \mathbf{w}\|^2 + \frac{1}{2}\int_{\Gamma_s} |\mathbf{z}^2|(\mathbf{v}_s \cdot \mathbf{n})^+ + \frac{1}{2}\int_{\Gamma_s} |\mathbf{z}^2|(\mathbf{z} \cdot \mathbf{n})^+ \\ \leq -\left(\lambda_\nu + K\alpha^2\right)\|\mathbf{v}\|^2,\end{aligned}$$

and hence

$$\frac{d}{dt}\|\mathbf{v}\|^2 + 2\left(\lambda_\nu + K\alpha^2\right)\|\mathbf{v}\|^2 \leq 0. \tag{3.39}$$

Multiplying (3.39) by $e^{2\sigma(t)}$, where

$$\sigma(t) = \lambda_\nu t + K\int_0^t \alpha^2(s)ds, \tag{3.40}$$

leads to

$$\frac{d}{dt}\left(\|\mathbf{v}\|^2 e^{2\sigma(t)}\right) \leq 0. \tag{3.41}$$

By integrating (3.41) from 0 to t we obtain the first a priori estimate

$$\|\mathbf{v}\| \leq \|\mathbf{v}(0)\| e^{-\sigma(t)}. \tag{3.42}$$

Further, from (3.38) we deduce

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \beta_\nu \alpha^2 \|\nabla \mathbf{w}\|^2 + \beta_\nu \|\nabla \mathbf{z}\|^2 \leq \lambda_\nu \|\mathbf{z}\|^2. \tag{3.43}$$

Let us estimate the term in the right hand side of (3.43). Since

$$\|\mathbf{z}\|^2 = \|\mathbf{v} - \alpha \mathbf{w}\|^2 \leq 2\|\mathbf{v}\|^2 + 2\alpha^2 \|\mathbf{w}\|^2,$$

using Lemma 2.5, we obtain

$$\lambda_\nu \|\mathbf{z}\|^2 \leq M_1 \|\mathbf{v}\|^2, \tag{3.44}$$

where $M_1 = 2\lambda_\nu \left(1 + C_e^2 \|\mathbf{w}\|^2\right)$. Further, employing (3.44) in (3.43) leads to

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \beta_\nu \left(\alpha^2 \|\nabla \mathbf{w}\|^2 + \|\nabla \mathbf{z}\|^2\right) \leq M_1 \|\mathbf{v}\|^2. \tag{3.45}$$

Integrating (3.45) from 0 to t, yields

$$\|\mathbf{v}\|^2 + 2\beta_\nu \int_0^t \left(\alpha^2\|\nabla\mathbf{w}\|^2 + \|\nabla\mathbf{z}\|^2\right) ds \;\leq\; \|\mathbf{v}(0)\|^2 + 2M_1 \int_0^t \|\mathbf{v}\|^2 ds, \tag{3.46}$$

obtaining $e^{-\sigma(t)} \leq e^{-\lambda_\nu t}$ from (3.40), and integrating (3.42) from 0 to t, leads to

$$\begin{aligned} 2\beta_\nu \int_0^t \left(\alpha^2\|\nabla\mathbf{w}\|^2 + \|\nabla\mathbf{z}\|^2\right) ds &\leq \left(1 + \frac{M_1}{\lambda_\nu} - \frac{M_1}{\lambda_\nu} e^{-2\lambda_\nu t}\right) \|\mathbf{v}(0)\|^2, \\ &\leq \left(1 + \frac{M_1}{\lambda_\nu}\right) \|\mathbf{v}(0)\|^2 = \left(3 + 2C_e^2\|\mathbf{w}\|^2\right) \|\mathbf{v}(0)\|^2 \end{aligned}$$

and hence, we obtain the second a priori estimate

$$\int_0^t \left(\alpha^2\|\nabla\mathbf{w}\|^2 + \|\nabla\mathbf{z}\|^2\right) ds \leq \left(\frac{3 + 2C_e^2\|\mathbf{w}\|^2}{2\beta_\nu}\right) \|\mathbf{v}(0)\|^2. \tag{3.47}$$

Consequently,

$$\begin{aligned} \int_0^t \|\nabla\mathbf{v}\|^2 ds &= \int_0^t \left(\alpha^2\|\nabla\mathbf{w}\|^2 + \|\nabla\mathbf{z}\|^2 + 2\,\alpha\langle\nabla\mathbf{w}, \nabla\mathbf{z}\rangle\right) ds \\ &\leq 2\int_0^t \left(\alpha^2\|\nabla\mathbf{w}\|^2 + \|\nabla\mathbf{z}\|^2\right) ds. \end{aligned}$$

Therefore, we obtain the last a priori estimate

$$\int_0^t \|\nabla\mathbf{v}\|^2 ds \leq \left(\frac{3 + 2C_e^2\|\mathbf{w}\|^2}{\beta_\nu}\right) \|\mathbf{v}(0)\|^2. \tag{3.48}$$

4.2 The variational formulation

In this section the variational formulation of the coupled system is obtained. By integrating by parts in space the stabilization problem (3.9), a weak formulation is obtained which leads to the following definition :

Definition 4.1. *Let $T > 0$ an arbitrary number and $\mathbf{v}_0 \in \mathbf{H}(\Omega)$, under assumptions (3.32) and (3.34), we shall say that $(\mathbf{v}, \alpha)$ is a weak solution of (3.9) on $[0, T)$ if*

(i) $\mathbf{v} \in [L^\infty(0, T; \mathbf{H}(\Omega)) \cap L^2(0, T; \mathbf{V}(\Omega))]$,

(ii) $\alpha \in L^\infty(0, T)$ *such that* $\mathbf{v}(t, \mathbf{x}) = \alpha(t)\mathbf{g}(\mathbf{x})$ *on* Γ_e*,*

(iii) $\forall \widetilde{\mathbf{v}} = \widetilde{\mathbf{z}} + \widetilde{\alpha}\mathbf{w} \in W(Q)$, *the following variational formulation is satisfied*

$$\begin{cases} (a) \quad \langle d_t \mathbf{v}, \widetilde{\mathbf{v}} \rangle + \nu a(\mathbf{v}, \widetilde{\mathbf{v}}) + b(\mathbf{v}, \mathbf{v}_s, \widetilde{\mathbf{v}}) + b(\mathbf{v}_s, \mathbf{v}, \widetilde{\mathbf{v}}) + b(\mathbf{v}, \mathbf{v}, \widetilde{\mathbf{v}}) = \displaystyle\int_{\Gamma_s} \mathbf{u}_s \cdot \widetilde{\mathbf{z}} + \widetilde{\alpha}\mathcal{F}(\mathbf{v}, \alpha), \\ (b) \quad \left(\displaystyle\int_\Omega \mathbf{v} \cdot \widetilde{\mathbf{v}} \, d\mathbf{x} \right)(t = 0) = \displaystyle\int_\Omega \mathbf{v}_0 \cdot \widetilde{\mathbf{v}} \, d\mathbf{x}. \end{cases} \tag{3.49}$$

Note that the initial condition (3.49-b) makes sense because for any solution $\mathbf{v}$ *of (3.49-a), function* $t \longrightarrow \int_\Omega \mathbf{v}(t) \cdot \widetilde{\mathbf{v}} \, d\mathbf{x}$ *is continuous (see [14] Corollaire II.4.2).*

The main achievement of this paper, is the following boundary stabilization result.

Theorem 4.2. *Assume that the steady-state* $\mathbf{v}_s$ *solution of (3.1) satisfies*

$$\beta_\nu = \nu - M_p \|\nabla \mathbf{v}_s\| > 0, \tag{3.50}$$

where M_p *is defined in (3.37). Assume that the initial condition* $\mathbf{v}_0$ *and the profile* $\mathbf{g}$ *satisfy*

$$\mathbf{v}_0 \in \mathbf{H}(\Omega), \quad (\mathbf{v}_0 \cdot \mathbf{n})\, \mathbf{n} \in \mathbf{H}^{1/2}(\Gamma_e), \tag{3.51}$$

$$\mathbf{g} \in \mathbf{V}^{1/2}(\Gamma_e) \quad \text{and} \quad \alpha_0 \, \mathbf{g} \cdot \mathbf{n} = \mathbf{v}_0 \cdot \mathbf{n} \ \text{ on } \ \Gamma_e \ \text{ with } \ \mathbf{g} \cdot \mathbf{n} \neq 0, \, \alpha_0 \in \mathbb{R}. \tag{3.52}$$

For arbitrary initial data $\mathbf{v}_0$ *satisfying (3.51), there exists a solution* $(\mathbf{v}, \alpha)$ *in the sense of definition 4.1, and a distribution* p *on* Q *such that (3.9) holds. Moreover, there exists a positive constant* σ *such that* $\mathbf{v}$ *satisfies*

$$\|\mathbf{v}\| \leq \|\mathbf{v}_0\| \exp \left(-\sigma t - K \int_0^t \alpha^2(s) ds \right), \tag{3.53}$$

where $K > 0$ *is a prescribed constant. Furthermore*

$$\int_0^T \|\nabla \mathbf{v}\|^2 \leq C_\nu \|\mathbf{v}_0\|^2, \tag{3.54}$$

where the constant C_ν *depends on* ν.

Remark 4.3. *With the condition (3.50), the equilibrium state* $\mathbf{v}_s$ *in (3.1) is naturally stable in the sense that the system (3.49) stabilizes by itself when* $\alpha \equiv 0$ *and* $\mathbf{z} \equiv 0$ *on* Γ_s*. This explains why the choice of the initial perturbation* $\mathbf{v}_0$*, in Theorem 4.2, is arbitrary. However, when* $\mathbf{z} \equiv 0$ *on* Γ_s*, as shown in Proposition 3.1, the control* α *is not identically zero as soon as the initial perturbation* $\mathbf{v}_0$ *and the profile* $\mathbf{g}$ *satisfy (2.48)-(2.49) with* $\mathbf{v}_0 \cdot \mathbf{n} \neq 0$*. The theoretical case* $\mathbf{v}_0 \cdot \mathbf{n} = 0$ *remains an open question.*

The proof of Theorem 4.2 is given at the end of this section, after Lemmas 4.4, 4.6 and 4.8 are established. In a first step a sequence of approximate solutions using a Galerkin method is built. A compactness result obtained in [25] then allows us to pass to the limit in the system satisfied by the approximated solutions.

4.3 The Galerkin Method

For all $m \in \mathbb{N}$, the space W_m is defined as :

$$W_m = span(\{\mathbf{w}_0, \mathbf{w}_1, \cdots, \mathbf{w}_m\}),$$

where $\mathbf{w}_0 = \mathbf{w}$ and $\mathbf{w}_i = \mathbf{z}_i,\ i = 1, 2, 3, \cdots, m$. Then for $(\mathbf{v}_m, \phi_{0m}) \in W_m$, we write $\mathbf{v}_m$ in the form $\mathbf{v}_m = \sum_{i=0}^{m} \phi_{im}\mathbf{w}_i$ and we define the following finite-dimensional problem

$$\begin{cases} (a) \quad \langle d_t\mathbf{v}_m, \mathbf{w}_j\rangle + \nu a(\mathbf{v}_m, \mathbf{w}_j) + b(\mathbf{v}_m, \mathbf{v}_s, \mathbf{w}_j) + b(\mathbf{v}_s, \mathbf{v}_m, \mathbf{w}_j) + b(\mathbf{v}_m, \mathbf{v}_m, \mathbf{w}_j) \\ \quad = \delta_{0j}\mathcal{F}(\mathbf{v}_m, \phi_{0m}) - \dfrac{1}{2}\displaystyle\int_{\Gamma_s}(\mathbf{z}_m\cdot\mathbf{w}_j)(\mathbf{v}_s\cdot\mathbf{n})^- - \dfrac{1}{2}\displaystyle\int_{\Gamma_s}(\mathbf{z}_m\cdot\mathbf{w}_j)(\mathbf{z}_m\cdot\mathbf{n})^-, \\ (b) \quad \langle\mathbf{v}_m(0) - \mathbf{v}_0, \mathbf{w}_j\rangle = 0,\ \text{for}\ j = 0, 1, 2, \cdots, m. \end{cases} \tag{3.55}$$

where $\mathbf{z}_m = \sum_{i=1}^{m} \phi_{im}\mathbf{w}_i$, δ_{0j} is the Kronecker symbol and

$$\begin{aligned} \mathcal{F}(\mathbf{v}_m, \phi_{0m}) \quad &= \quad a_e\phi_{0m}^2 + b_e\phi_{0m} - \lambda_\nu\left(\phi_{0m}\|\mathbf{w}\|^2 + 2\langle\mathbf{w}, \mathbf{z}_m\rangle\right) \\ &\qquad + 2\,\beta_\nu\langle\nabla\mathbf{w}, \nabla\mathbf{z}_m\rangle - K\phi_{0m}\|\mathbf{v}_m\|^2. \end{aligned} \tag{3.56}$$

Lemma 4.4. *The discrete problem* (3.55) *has a unique solution* $\mathbf{v}_m \in C^1(0, T_m; W_m)$. *Moreover the solution satisfies :*

$$\|\mathbf{v}_m\|_{L^\infty(0,T;\,\mathbf{L}^2(\Omega))} + \|\mathbf{v}_m\|_{L^2(0,T;\,\mathbf{H}^1(\Omega))} \leq C, \tag{3.57}$$

where C is a positive constant independent of m.

Proof. Classical results of nonlinear ODEs lead to the existence of the greatest T_m in $(0, T)$ such that the discrete problem (3.55) has a unique solution $\mathbf{v}_m \in C^1(0, T_m; W_m)$. Indeed, the resulting mass matrix defined as $M_{ij} = \langle\mathbf{w}_i, \mathbf{w}_j\rangle$ $(0 \leq i, j \leq m)$ is nonsingular. In order to show that T_m is independent of m, it is sufficient to verify the boundedness of the L^2-norm of $\mathbf{v}_m$ independently of m.

Multiplying (3.55-b) by $\phi_{jm}(0)$, and summing for $j = 0, 1, 2, \cdots, m$, leads to

$$\int_\Omega |\mathbf{v}_m(0)|^2 = \int_\Omega \mathbf{v}_0\cdot\mathbf{v}_m(0) \leq \frac{1}{2}\int_\Omega |\mathbf{v}_0|^2 + \frac{1}{2}\int_\Omega |\mathbf{v}_m(0)|^2,$$

and then

$$\|\mathbf{v}_m(0)\|^2 \leq \|\mathbf{v}_0\|^2. \tag{3.58}$$

Following the same procedure as for the derivation of the a priori estimates (3.42) and (3.48), and using (3.58) yields

$$\begin{cases} (a) \quad \|\mathbf{v}_m\| \leq \|\mathbf{v}_0\| \, e^{-\sigma(t)}, \\ (b) \quad \displaystyle\int_0^T \|\nabla \mathbf{v}_m\|^2 dt \leq C\|\mathbf{v}_0\|^2. \end{cases} \tag{3.59}$$

If $T_m < T$, then $\|\mathbf{v}_m\|$ should tend to $+\infty$ as $t \to T_m$ because of the explosion criteria. However, this does not happen since $\|\mathbf{v}_m\|$ is bounded independently of m in (3.59-a), and therefore $T_m = T$. □

A consequence of the inequality (3.57) is that $(\mathbf{v}_m)_m, m = 0, 1, 2, \cdots$, is bounded in $L^2(0, T; \mathbf{V}(\Omega))$ and $L^\infty(0, T; \mathbf{H}(\Omega))$. Therefore, for a subsequence of $\mathbf{v}_m$ (still denoted by $\mathbf{v}_m$), inequality (3.57) yields the following weak convergences as m tends to ∞ :

$$\begin{cases} \mathbf{v}_m \rightharpoonup \mathbf{v} \text{ weakly in } L^2(0, T; \mathbf{V}(\Omega)), \\ \mathbf{v}_m \rightharpoonup \mathbf{v} \text{ weakly* in } L^\infty(0, T; \mathbf{H}(\Omega)). \end{cases} \tag{3.60}$$

Nervertheless, the convergences in (3.60) are not sufficient to pass to the limit in the weak formulation (3.55), because of the presence of the convection term. Consequently, we need to obtain additional bounds in order to utilize the compactness theory on the sequence of approximated solution $(\mathbf{v}_m)_m, m = 0, 1, 2, \cdots$.

4.4 Additional bounds

Let us assume that B_0, B and B_1 are three Hilbert spaces such that $B_0 \subset B \subset B_1$. If $\mathbf{v} : \mathbb{R} \to B_1$ is a function, we denote by $\widehat{\mathbf{v}}$ its Fourier transform

$$\widehat{\mathbf{v}}(\tau) = \int_{-\infty}^{+\infty} e^{-2i\pi t\tau} \mathbf{v}(t) dt.$$

Recall the following identity about the Fourier transform of differential operators

$$\widehat{D_t^\gamma \mathbf{v}(\tau)} = (2i\pi\tau)^\gamma \widehat{\mathbf{v}}(\tau),$$

for a given $\gamma > 0$, and let us define the space

$$H^\gamma(\mathbb{R}; B_0, B_1) = \{\mathbf{u} \in L^2(\mathbb{R}, B_0), D_t^\gamma \mathbf{u} \in L^2(\mathbb{R}, B_1)\}.$$

The space $H^\gamma(\mathbb{R}; B_0, B_1)$ is endowed with the norm

$$\|\mathbf{v}\|_{H^\gamma(\mathbb{R};B_0,B_1)} = (\|\mathbf{v}\|^2_{L^2(\mathbb{R};B_0)} + \||\tau|^\gamma \widehat{\mathbf{v}}\|^2_{L^2(\mathbb{R};B_1)})^{\frac{1}{2}}.$$

We also define $H^\gamma(0, T; B_0, B_1)$, as the space of functions obtained by restriction to $[0, T]$ of functions of $H^\gamma(\mathbb{R}; B_0, B_1)$. Further, we recall the following result [25] :

Lemma 4.5. *Let B_0, B and B_1 be three Hilbert spaces such that $B_0 \subset B \subset B_1$ and B_0 is compactly embedded in B. Then for all $\gamma > 0$, the injection $H^\gamma(0, T; B_0, B_1) \to L^2(0, T; B)$ is compact.*

Lemma 4.5 is used later with : $B_0 = \mathbf{V}(\Omega)$, $B = \mathbf{H}(\Omega)$, $B_1 = \mathbf{H}(\Omega)$ and $0 < \gamma < \frac{1}{4}$.

The main result of Section 4.4 is furnished by the following lemma :

Lemma 4.6. *For $0 < \gamma < \frac{1}{4}$, the sequence $\mathbf{v}_m$ is bounded in $H^\gamma(0, T; \mathbf{V}(\Omega), \mathbf{H}(\Omega))$.*

Proof. Since we already know that $\mathbf{v}_m$ is uniformly bounded in $L^2(0, T, \mathbf{V}(\Omega))$ from (3.57), it remains to prove that

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \|\widehat{\mathbf{v}}_m\|^2 \leq C. \tag{3.61}$$

We denote by $\overline{\mathbf{v}}_m$ the extension of $\mathbf{v}_m$ by zero for $t < 0$ and $t > T$, and $\widehat{\mathbf{v}}_m$ the Fourier transform of $\overline{\mathbf{v}}_m$ with respect to time. Classical results show that since $\overline{\mathbf{v}}_m$ has two discontinuities at 0 and T, in the distributional sense, the derivative of $\overline{\mathbf{v}}_m$ is given by

$$\frac{d}{dt}\overline{\mathbf{v}}_m = \overline{\mathbf{u}}_m + \mathbf{v}_m(0)\delta_0 - \mathbf{v}_m(T)\delta_T, \tag{3.62}$$

where δ_0, δ_T are Dirac distributions at 0 and T, and

$$\overline{\mathbf{u}}_m = \mathbf{v}'_m = \text{the derivative of } \mathbf{v}_m \text{ on } [0, T].$$

By taking the Fourier transform of (3.62) we obtain

$$2i\pi\tau\widehat{\mathbf{v}}_m(\tau) = \widehat{\mathbf{u}}_m(\tau) + \mathbf{v}_m(0) - \mathbf{v}_m(T)e^{-2i\pi\tau T},$$

where $\widehat{\mathbf{v}}_m$ and $\widehat{\mathbf{u}}_m$ denote the Fourier transforms of $\overline{\mathbf{v}}_m$ and $\overline{\mathbf{u}}_m$ respectively.

The finite-dimensional problem (3.55) is now considered for all time independent test function $\widetilde{\mathbf{v}} = \widetilde{\mathbf{z}} + \widetilde{\alpha}\mathbf{w}_0 \in W_m$, and $\mathbf{v}_m$ is replaced by $\overline{\mathbf{v}}_m$. This leads to

$$\begin{aligned} &\int_\Omega \frac{\partial \overline{\mathbf{v}}_m}{\partial t} \cdot \widetilde{\mathbf{v}} + \nu \int_\Omega \nabla \overline{\mathbf{v}}_m : \nabla \widetilde{\mathbf{v}} + \int_\Omega G_m \cdot \widetilde{\mathbf{v}} + \int_\Omega G_m^s \cdot \widetilde{\mathbf{v}} + \int_{\Gamma_s} Z_m \cdot \widetilde{\mathbf{v}} + \int_\Omega (\mathbf{v}_s \cdot \nabla)\overline{\mathbf{v}}_m \cdot \widetilde{\mathbf{v}} \\ &= -\frac{1}{2} \int_{\Gamma_s} (\overline{\mathbf{z}}_m \cdot \widetilde{\mathbf{z}})(\mathbf{v}_s \cdot \mathbf{n})^- + \widetilde{\alpha} H_m + \int_\Omega \overline{\mathbf{v}}_m(0) \cdot \widetilde{\mathbf{v}} \delta_0 - \int_\Omega \overline{\mathbf{v}}_m(T) \cdot \widetilde{\mathbf{v}} \delta_T, \end{aligned} \tag{3.63}$$

where $H_m = \mathcal{F}(\overline{\mathbf{v}}_m, \overline{\phi}_{0m})$ is defined to be the extension of $\mathcal{F}(\mathbf{v}_m, \phi_{0m})$ by zero for $t < 0$ and $t > T$, and $G_m = (\overline{\mathbf{v}}_m \cdot \nabla)\overline{\mathbf{v}}_m$, $G_m^s = (\overline{\mathbf{v}}_m \cdot \nabla)\mathbf{v}_s$ and $Z_m = \frac{1}{2}\overline{\mathbf{z}}_m(\overline{\mathbf{z}}_m \cdot \mathbf{n})^-$. Taking the Fourier transform of (3.63) yields

$$\begin{aligned} 2i\pi\tau &\int_\Omega \widehat{\mathbf{v}}_m(\tau) \cdot \widetilde{\mathbf{v}} + \nu \int_\Omega \nabla \widehat{\mathbf{v}}_m(\tau) : \nabla \widetilde{\mathbf{v}} + \int_\Omega \widehat{G}_m(\tau) \cdot \widetilde{\mathbf{v}} + \int_\Omega \widehat{G}_m^s(\tau) \cdot \widetilde{\mathbf{v}} \\ &+ \int_{\Gamma_s} \widehat{Z}_m(\tau) \cdot \widetilde{\mathbf{v}} + \int_\Omega (\mathbf{v}_s \cdot \nabla)\widehat{\mathbf{v}}_m \cdot \widetilde{\mathbf{v}} = -\frac{1}{2} \int_{\Gamma_s} (\widehat{\mathbf{z}}_m \cdot \widetilde{\mathbf{z}})(\mathbf{v}_s \cdot \mathbf{n})^- + \widetilde{\alpha}\widehat{H}_m \\ &\quad + \int_\Omega \overline{\mathbf{v}}_m(0) \cdot \widetilde{\mathbf{v}} - \int_\Omega \overline{\mathbf{v}}_m(T) \cdot \widetilde{\mathbf{v}} e^{-2i\pi\tau T}, \end{aligned} \tag{3.64}$$

where $\widehat{G}_m$, $\widehat{G}_m^s$, $\widehat{Z}_m$ and $\widehat{H}_m$ are the Fourier transforms (with respect to time) of G_m, G_m^s, Z_m and H_m, respectively.

We now take $(\widetilde{\mathbf{v}}, \widetilde{\alpha}) = (\widehat{\mathbf{v}}_m, \widehat{\phi}_{0m}) \in W_m$ in (3.64), and we obtain

$$\begin{aligned} 2i\pi\tau &\int_\Omega |\widehat{\mathbf{v}}_m(\tau)|^2 + \nu \int_\Omega \nabla \widehat{\mathbf{v}}_m(\tau) : \nabla \widehat{\mathbf{v}}_m(\tau) + \int_\Omega \widehat{G}_m(\tau) \cdot \widehat{\mathbf{v}}_m(\tau) + \int_\Omega \widehat{G}_m^s(\tau) \cdot \widehat{\mathbf{v}}_m(\tau) \\ &+ \int_{\Gamma_s} \widehat{Z}_m(\tau) \cdot \widehat{\mathbf{v}}_m + \int_\Omega (\mathbf{v}_s \cdot \nabla)\widehat{\mathbf{v}}_m \cdot \widehat{\mathbf{v}}_m(\tau) = -\frac{1}{2} \int_{\Gamma_s} |\widehat{\mathbf{z}}_m|^2 (\mathbf{v}_s \cdot \mathbf{n})^- \\ &\quad + \widehat{\phi}_{0m}\widehat{H}_m + \int_\Omega \overline{\mathbf{v}}_m(0) \cdot \widehat{\mathbf{v}}_m(\tau) - \int_\Omega \overline{\mathbf{v}}_m(T) \cdot \widehat{\mathbf{v}}_m(\tau) e^{-2i\pi\tau T}. \end{aligned} \tag{3.65}$$

According to (3.56), we have

$$\widehat{\phi}_{0m}\widehat{H}_m = \widehat{\phi}_{0m}\widehat{Y}_m + b_e(\widehat{\phi}_{0m})^2 - \lambda_\nu \widehat{\phi}_{0m} \left(\widehat{\phi}_{0m} \|\mathbf{w}\|^2 + 2\langle \mathbf{w}, \widehat{\mathbf{z}}_m \rangle \right) \tag{3.66}$$

where $\widehat{Y}_m$ is the Fourier transform of

$$Y_m = a_e \overline{\phi}_{0m}^2 + 2\,\beta_\nu \langle \nabla \mathbf{w}, \nabla \overline{\mathbf{z}}_m \rangle - K \overline{\phi}_{0m} \|\overline{\mathbf{v}}_m\|^2, \tag{3.67}$$

and rewritting the last term of (3.66) leads to

$$\widehat{\phi}_{0m}\widehat{H}_m = \widehat{\phi}_{0m}\widehat{Y}_m + b_e(\widehat{\phi}_{0m})^2 - \lambda_\nu \left(\|\widehat{\mathbf{v}}_m(\tau)\|^2 - \|\widehat{\mathbf{z}}_m(\tau)\|^2 \right). \tag{3.68}$$

Due to (3.27), we obtain

$$\int_\Omega (\mathbf{v}_s\nabla)\widehat{\mathbf{v}}_m \cdot \widehat{\mathbf{v}}_m(\tau) = b_e(\widehat{\phi}_{0m})^2 + \frac{1}{2}\int_{\Gamma_s} |\widehat{\mathbf{z}}_m|^2(\mathbf{v}_s\cdot\mathbf{n})\,d\zeta. \tag{3.69}$$

and using (3.68)-(3.69) in (3.65) yields

$$\begin{aligned}
2i\pi\tau\int_\Omega |\widehat{\mathbf{v}}_m(\tau)|^2 + \nu\int_\Omega \nabla\widehat{\mathbf{v}}_m(\tau) : \nabla\widehat{\mathbf{v}}_m(\tau) + \int_\Omega \widehat{G}_m(\tau)\cdot\widehat{\mathbf{v}}_m(\tau) + \int_\Omega \widehat{G}^s_m(\tau)\cdot\widehat{\mathbf{v}}_m(\tau)\\
+\int_{\Gamma_s}\widehat{Z}_m(\tau)\cdot\widehat{\mathbf{v}}_m + \frac{1}{2}\int_{\Gamma_s}|\widehat{\mathbf{z}}_m|^2(\mathbf{v}_s\cdot\mathbf{n})^+ = \widehat{\phi}_{0m}\widehat{Y}_m - \lambda_\nu\left(\|\widehat{\mathbf{v}}_m(\tau)\|^2 - \|\widehat{\mathbf{z}}_m(\tau)\|^2\right)\\
+\int_\Omega \overline{\mathbf{v}}_m(0)\cdot\widehat{\mathbf{v}}_m(\tau) - \int_\Omega \overline{\mathbf{v}}_m(T)\cdot\widehat{\mathbf{v}}_m(\tau)e^{-2i\pi\tau T}.
\end{aligned}\tag{3.70}$$

Thanks to Remark 2.6 of Lemma 2.5, we have

$$|\widehat{\phi}_{0m}(\tau)| \le C_e\|\widehat{\mathbf{v}}_m(\tau)\| \tag{3.71}$$

and due to the trace theorem, there exists a positive constant C such that $\|\mathbf{u}\|_{\mathbf{L}^2(\Gamma)} \le C\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}$, $\forall\mathbf{u}\in\mathbf{H}^1(\Omega)$ and hence

$$\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{L}^2(\Gamma_s)} \le C\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}. \tag{3.72}$$

Thus, taking the imaginary part of (3.70) and using (3.71)-(3.72) and Remark 2.1, leads to

$$\begin{aligned}
|\tau|\|\widehat{\mathbf{v}}_m(\tau)\|^2 \le C\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{V}(\Omega)}\big(\|\widehat{G}_m(\tau)\|_{\mathbf{V}'(\Omega)} + \|\widehat{G}^s_m(\tau)\|_{\mathbf{V}'(\Omega)} + \|\widehat{Z}_m(\tau)\|_{\mathbf{L}^2(\Gamma_s)}\big)\\
+ C\,\|\widehat{\mathbf{v}}_m(\tau)\|\big(|\widehat{Y}_m(\tau)| + \|\overline{\mathbf{v}}_m(0)\| + \|\overline{\mathbf{v}}_m(T)\|\big).
\end{aligned}\tag{3.73}$$

Note that in the sequel, C stands for different positive constants. We now prove that each term lying in the right hand side of (3.73) is bounded.

First, we have

$$\|G_m\|_{\mathbf{V}'(\Omega)} \le c_1\|\mathbf{v}_m\|^2_{\mathbf{H}^1(\Omega)} \quad\text{and}\quad \|G^s_m\|_{\mathbf{V}'(\Omega)} \le C\|\mathbf{v}_m\|_{\mathbf{H}^1(\Omega)},$$

and thanks to the energy estimates (3.59) satisfied by $\mathbf{v}_m$, the quantities G_m and G^s_m remain bounded in $L^1(\mathbb{R};\mathbf{V}'(\Omega))$, and sequences $\widehat{G}_m$, $\widehat{G}^s_m$ are bounded in $L^\infty(\mathbb{R};\mathbf{V}'(\Omega))$ i.e.

$$\sup_{\tau\in\mathbb{R}}\left(\|\widehat{G}_m(\tau)\|_{\mathbf{V}'(\Omega)} + \|\widehat{G}^s_m(\tau)\|_{\mathbf{V}'(\Omega)}\right) \le C.$$

Thanks to Hölder inequality and the trace theorem in [14, P 249, Théorème V.2.2], we have

$$2\|Z_m\|_{\mathbf{L}^2(\Gamma_s)} = \|\overline{\mathbf{z}}_m(\overline{\mathbf{z}}_m \cdot \mathbf{n})^-\|_{\mathbf{L}^2(\Gamma_s)} \quad \leq \quad \|\mathbf{z}_m\|_{\mathbf{L}^4(\Gamma_s)}\|\mathbf{z}_m\|_{\mathbf{L}^4(\Gamma_s)} \leq C\|\mathbf{z}_m\|^2_{\mathbf{H}^1(\Omega)}. \tag{3.74}$$

Following the same procedure as for the derivation of the a priori estimates (3.47), and using (3.58) we have $\mathbf{z}_m \in L^2(0,T;\mathbf{H}^1(\Omega))$. Further, by using (3.74) we show that Z_m is bounded in $L^1(\mathbb{R};\mathbf{L}^2(\Gamma_s))$ and hence $\widehat{Z}_m$ is bounded in $L^\infty(\mathbb{R};\mathbf{L}^2(\Gamma_s))$.

We now show that Y_m is bounded in $L^1(\mathbb{R})$. From (3.67), thanks to Hölder inequality and Lemma 2.5, we have

$$|Y_m| \leq a_e\phi^2_{0m} + 2\,\beta_\nu\|\nabla\mathbf{w}\|_{\mathbf{H}^1(\Omega)}\|\nabla\mathbf{z}_m\|_{\mathbf{H}^1(\Omega)} + KC_e\|\mathbf{v}_m\|^3$$

and since $\mathbf{z}_m \in L^2(0,T;\mathbf{H}^1(\Omega))$, according to (3.59) we show that $Y_m \in L^1(\mathbb{R})$ and hence $\widehat{Y}_m$ is bounded in $L^\infty(\mathbb{R})$ with

$$\sup_{\tau\in\mathbb{R}} |\widehat{Y}_m(\tau)| \leq C.$$

Finally, it remains to show that the two last terms in the right hand side of (3.73) are bounded. Thanks to the energy estimates (3.59), we have $\|\mathbf{v}_m(T)\| \leq C$ and $\|\mathbf{v}_m(0)\| \leq C$. Inequation (3.73) then reduces to

$$|\tau|\|\widehat{\mathbf{v}}_m(\tau)\|^2 \quad \leq \quad C(\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)} + \|\widehat{\mathbf{v}}_m(\tau)\|). \tag{3.75}$$

Therefore, we obtain the following inequality

$$|\tau|\|\widehat{\mathbf{v}}_m(\tau)\|^2 \quad \leq \quad C\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}, \tag{3.76}$$

where C stands for different positive constants.

From [31, Chapter 3, Section 3.2, page 194] we have

$$|\tau|^{2\gamma} \leq c(\gamma)\frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}, \ \forall\tau\in\mathbb{R}, \ \text{with } 0<\gamma<1/4,$$

and consequently, we deduce

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma}\|\widehat{\mathbf{v}}_m(\tau)\|^2 \leq c(\gamma)\int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|^2}{1+|\tau|^{1-2\gamma}} + c(\gamma)\int_{-\infty}^{+\infty} \frac{|\tau|\|\widehat{\mathbf{v}}_m(\tau)\|^2}{1+|\tau|^{1-2\gamma}}. \tag{3.77}$$

For the first integral in the right hand side of (3.77) the Poincaré inequality is used, and thanks to (3.76) the second integral in the right hand side of (3.77) is rewritten. This leads to

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma}\|\widehat{\mathbf{v}}_m(\tau)\|^2 \le c_3(\gamma)\int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|^2_{\mathbf{H}^1(\Omega)}}{1+|\tau|^{1-2\gamma}} + c_4(\gamma)\int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}}{1+|\tau|^{1-2\gamma}}$$
$$\le c_3(\gamma)\int_{-\infty}^{+\infty} \|\widehat{\mathbf{v}}_m(\tau)\|^2_{\mathbf{H}^1(\Omega)} + c_4(\gamma)\int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}}{1+|\tau|^{1-2\gamma}}. \tag{3.78}$$

The second integral in the right hand side of (3.78) satisfies

$$\int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}}{1+|\tau|^{1-2\gamma}} \le \left(\int_{-\infty}^{+\infty} \frac{1}{(1+|\tau|^{1-2\gamma})^2}\right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} \|\widehat{\mathbf{v}}_m(\tau)\|^2_{\mathbf{H}^1(\Omega)}\right)^{\frac{1}{2}}, \tag{3.79}$$

and the first integral in the right hand side of (3.79) is convergent for any $0<\gamma<\frac{1}{4}$. On the other hand, using the Parseval equality leads to

$$\int_{-\infty}^{+\infty} \|\widehat{\mathbf{v}}_m(\tau)\|^2_{\mathbf{H}^1(\Omega)} d\tau = \int_0^T \|\mathbf{v}_m(t)\|^2_{\mathbf{H}^1(\Omega)} dt \le C,$$

which implies that the sequence $\mathbf{v}_m$ is bounded in $H^\gamma(0,T;\mathbf{V}(\Omega),\mathbf{H}(\Omega))$. □

Finally, by applying Lemmas 4.5 and 4.6, we conclude that there exists a subsequence of $(\mathbf{v}_m)_{m\in\mathbb{N}}$ which converges strongly in $L^2(0,T,\mathbf{H}(\Omega))$.

4.5 Passage to the limit

The compactness result obtained in the previous section implies the following strong convergence (at least for a subsequence of $\mathbf{v}_m$ still denoted by $\mathbf{v}_m$)

$$\mathbf{v}_m \longrightarrow \mathbf{v} \text{ strongly in } L^2(0,T;\mathbf{L}^2(\Omega)).$$

Such a convergence result together with (3.60) enables us to pass to the limit in the following weak formulation, obtained from (3.55) after multiplication by $\varphi\in\mathcal{D}([0,T))$ and integration by parts with respect to time, i.e.

$$-\int_0^T\int_\Omega \mathbf{v}_m\cdot\widetilde{\mathbf{v}}_j\varphi'(t) + \nu\int_0^T\int_\Omega \nabla\mathbf{v}_m : \nabla\widetilde{\mathbf{v}}_j\varphi(t) + \int_0^T\int_\Omega (\mathbf{v}_m\cdot\nabla\mathbf{v}_m)\cdot\widetilde{\mathbf{v}}_j\varphi(t)$$
$$+\int_0^T\int_\Omega (\mathbf{v}_m\cdot\nabla\mathbf{v}_s)\cdot\widetilde{\mathbf{v}}_j\varphi(t) + \int_0^T\int_\Omega (\mathbf{v}_s\cdot\nabla\mathbf{v}_m)\cdot\widetilde{\mathbf{v}}_j\varphi(t) - \varphi(0)\int_\Omega \mathbf{v}_m(0)\cdot\widetilde{\mathbf{v}}_j$$
$$= -\frac{1}{2}\int_0^T\int_{\Gamma_s}(\mathbf{z}_m\cdot\widetilde{\mathbf{v}}_j)(\mathbf{v}_s\cdot\mathbf{n})^-\varphi(t) - \frac{1}{2}\int_0^T\int_{\Gamma_s}(\mathbf{z}_m\cdot\widetilde{\mathbf{v}}_j)(\mathbf{z}_m\cdot\mathbf{n})^-\varphi(t)$$
$$+\int_0^T \widetilde{\alpha}_j\delta_{0j}\mathcal{F}(\mathbf{v}_m,\phi_{0m})\varphi(t), \quad \forall\,\widetilde{\mathbf{v}}_j = \widetilde{\alpha}_j\mathbf{w}_j. \tag{3.80}$$

As a first step the integrals in the left hand side of (3.80) are examined.

The weak convergences (3.60) yield

$$\begin{aligned}
\int_0^T \int_\Omega \mathbf{v}_m \cdot \widetilde{\mathbf{v}}_j \varphi'(t) &\xrightarrow[m\to+\infty]{} \int_0^T \int_\Omega \mathbf{v} \cdot \widetilde{\mathbf{v}}_j \varphi'(t),\\
\int_0^T \int_\Omega \nabla \mathbf{v}_m : \nabla \widetilde{\mathbf{v}}_j \varphi(t) &\xrightarrow[m\to+\infty]{} \int_0^T \int_\Omega \nabla \mathbf{v} : \nabla \widetilde{\mathbf{v}}_j \varphi(t),\\
\int_0^T \int_\Omega (\mathbf{v}_m \cdot \nabla \mathbf{v}_s) \cdot \widetilde{\mathbf{v}}_j \varphi(t) &\xrightarrow[m\to+\infty]{} \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}_s) \cdot \widetilde{\mathbf{v}}_j \varphi(t),\\
\int_0^T \int_\Omega (\mathbf{v}_s \cdot \nabla \mathbf{v}_m) \cdot \widetilde{\mathbf{v}}_j \varphi(t) &\xrightarrow[m\to+\infty]{} \int_0^T \int_\Omega (\mathbf{v}_s \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}}_j \varphi(t)
\end{aligned}$$

for the linear terms in the left hand side of (3.80). Further, by using (3.55-b), we have

$$\int_\Omega \mathbf{v}_m(0)\widetilde{\mathbf{v}}_j = \int_\Omega \mathbf{v}_0 \widetilde{\mathbf{v}}_j.$$

Since $\mathbf{v}_m$ converges to $\mathbf{v}$ weakly in $L^2(0,T;\mathbf{V}(\Omega))$, and strongly in $L^2(0,T;\mathbf{L}^2(\Omega))$, we can pass to the limit in the nonlinear term to obtain

$$\int_0^T \int_\Omega (\mathbf{v}_m \cdot \nabla \mathbf{v}_m) \cdot \widetilde{\mathbf{v}}_j \varphi(t) \xrightarrow[m\to+\infty]{} \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}}_j \varphi(t).$$

As a second step the boundary terms in the right hand side of (3.80) are examined. Since $\mathbf{z}_m \in L^2(0,T;\mathbf{H}^1(\Omega))$, thanks to [14, Proposition V.2.5], we obtain

$$\begin{aligned}
\mathbf{z}_m &\to \mathbf{z} \text{ strongly in } L^2(0,T;\mathbf{L}^2(\Gamma))\\
\mathbf{z}_m(\mathbf{z}_m\cdot\mathbf{n})^- + \mathbf{z}_m(\mathbf{v}_s\cdot\mathbf{n})^- &\rightharpoonup \mathbf{z}(\mathbf{z}\cdot\mathbf{n})^- + \mathbf{z}(\mathbf{v}_s\cdot\mathbf{n})^- \text{ weakly in } L^{\frac{4}{3}}(0,T;\mathbf{L}^{\frac{4}{3}}(\Gamma)).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_0^T \int_{\Gamma_s} (\mathbf{z}_m \cdot \widetilde{\mathbf{v}}_j)(\mathbf{v}_s \cdot \mathbf{n})^- \varphi(t) &\xrightarrow[m\to+\infty]{} \int_0^T \int_{\Gamma_s} (\mathbf{z} \cdot \widetilde{\mathbf{v}}_j)(\mathbf{v}_s \cdot \mathbf{n})^- \varphi(t),\\
\int_0^T \int_{\Gamma_s} (\mathbf{z}_m \cdot \widetilde{\mathbf{v}}_j)(\mathbf{z}_m \cdot \mathbf{n})^- \varphi(t) &\xrightarrow[m\to+\infty]{} \int_0^T \int_{\Gamma_s} (\mathbf{z} \cdot \widetilde{\mathbf{v}}_j)(\mathbf{z} \cdot \mathbf{n})^- \varphi(t).
\end{aligned}$$

As a last step, thanks to (3.56), we prove the convergence of the last integral lying in the right hand side of (3.80), which reads

$$\int_0^T \widetilde{\alpha}_j \mathcal{F}(\mathbf{v}_m, \phi_{0m})\varphi(t) = a_e \int_0^T \phi_{0m}^2 \widetilde{\alpha}_j \varphi(t) + b_e \int_0^T \phi_{0m} \widetilde{\alpha}_j \varphi(t)$$
$$- \lambda_\nu \int_0^T \big(\phi_{0m} \|\mathbf{w}\|^2 + 2\langle \mathbf{w}, \mathbf{z}_m \rangle \big) \widetilde{\alpha}_j \varphi(t) + 2\,\beta_\nu \int_0^T \langle \nabla \mathbf{w}, \nabla \mathbf{z}_m \rangle \widetilde{\alpha}_j \varphi(t)$$
$$- K \int_0^T \phi_{0m} \|\mathbf{v}_m\|^2 \widetilde{\alpha}_j \varphi(t). \tag{3.81}$$

By using Lemma 2.5 and according to (3.59-a), $\phi_{0m} \in L^\infty(0,T)$, and hence, for a subsequence of ϕ_{0m} (still denoted by ϕ_{0m})

$$\phi_{0m} \rightharpoonup \alpha \text{ weakly}^* \text{ in } L^\infty(0,T).$$

Let us notice that the convergence of $\mathbf{v}_m$ in $L^2([0,T] \times \Omega)$ implies the convergence of $\mathbf{v}_m$ in $L^1(0,T;\mathbf{L}^2(\Omega))$, and hence

$$\|\mathbf{v}_m\| \quad \longrightarrow \quad \|\mathbf{v}\| \text{ strongly in } L^1(0,T). \tag{3.82}$$

Further, due to Lemma 2.5, we obtain

$$|\phi_{0p} - \phi_{0q}| \leq C_e \|\mathbf{v}_p - \mathbf{v}_q\|, \quad \forall (\mathbf{v}_p, \phi_{0p}), (\mathbf{v}_q, \phi_{0q}) \in W_m.$$

Consequently, ϕ_{0m} is a Cauchy sequence in $L^1(0,T)$ and it converges to a limit ϕ_0 in $L^1(0,T)$ i.e.

$$\phi_{0m} \longrightarrow \phi_0 \text{ strongly in } L^1(0,T). \tag{3.83}$$

Therefore, we conclude that $\phi_0 = \alpha \in L^\infty(0,T)$ due to [14, Proposition II.1.26]. By using (3.57), the quantities $\|\mathbf{v}_m\|$ and ϕ_{0m} are bounded in $L^\infty(0,T)$ and in addition, from (3.82) and (3.83) we obtain for all $p \in]1,+\infty[$

$$\begin{aligned} \|\mathbf{v}_m\| \quad &\longrightarrow \quad \|\mathbf{v}\| \text{ strongly in } L^p(0,T), \\ \phi_{0m} \quad &\longrightarrow \quad \alpha \quad \text{strongly in } L^p(0,T). \end{aligned}$$

This is due to a result obtained in [14, Corollaire II.1.24] which states that if a sequence of functions converges strongly in $L^1(0,T)$ and weakly star in $L^\infty(0,T)$, then $\forall p \in]1,\infty[$ the sequence converges strongly in $L^p(0,T)$.

Finally, it now remains to pass to the limit in each term in the right hand side of (3.81). This leads to

$$\int_0^T \widetilde{\alpha}_j \mathcal{F}(\mathbf{v}_m, \phi_{0m})\varphi(t) \xrightarrow[m\to+\infty]{} \int_0^T \widetilde{\alpha}_j \mathcal{F}(\mathbf{v}, \alpha)\varphi(t),$$

where

$$\mathcal{F}(\mathbf{v}, \alpha) = a_e\alpha^2 + b_e\alpha - \lambda_\nu \left(\alpha\|\mathbf{w}\|^2 + 2\langle \mathbf{w}, \mathbf{z}\rangle\right) + 2\,\beta_\nu\langle\nabla\mathbf{w}, \nabla\mathbf{z}\rangle - K\alpha\|\mathbf{v}\|^2.$$

As a final step, passing to the limit in (3.80) yields

$$\begin{aligned}
&-\int_0^T\int_\Omega \mathbf{v}\cdot\widetilde{\mathbf{v}}_j\varphi'(t) + \nu\int_0^T\int_\Omega \nabla\mathbf{v} : \nabla\widetilde{\mathbf{v}}_j\varphi(t) + \int_0^T\int_\Omega(\mathbf{v}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}_j\varphi(t)\\
&\quad+\int_0^T\int_\Omega(\mathbf{v}\cdot\nabla\mathbf{v}_s)\cdot\widetilde{\mathbf{v}}_j\varphi(t) + \int_0^T\int_\Omega(\mathbf{v}_s\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}_j\varphi(t) - \int_\Omega \mathbf{v}_0\widetilde{\mathbf{v}}_j\varphi(0)\\
&\quad= -\frac{1}{2}\int_0^T\int_{\Gamma_s}(\mathbf{z}\cdot\widetilde{\mathbf{v}}_j)(\mathbf{v}_s\cdot\mathbf{n})^-\varphi(t) - \frac{1}{2}\int_0^T\int_{\Gamma_s}(\mathbf{z}\cdot\widetilde{\mathbf{v}}_j)(\mathbf{z}\cdot\mathbf{n})^-\varphi(t)\\
&\quad\quad+\int_0^T \widetilde{\alpha}_j\delta_{0j}\mathcal{F}(\mathbf{v}, \alpha)\varphi(t),
\end{aligned}\tag{3.84}$$

for all $\widetilde{\mathbf{v}}_j = \widetilde{\alpha}_j\mathbf{w}_j,\ j = 0, 1, 2, \cdots, m$. By linearity, equation (3.84) holds true for all $\widetilde{\mathbf{v}}$ combination of finite $\widetilde{\mathbf{v}}_j$ and by density, for any element of $W(Q)$.

In the remaining part of this paper, our purpose is first to retrieve the weak formulation (3.49) and secondly (in Section 4.6) to obtain the original system (3.9) including the initial condition.

By choosing $\varphi \in \mathcal{D}(]0, T[)$ in (3.84) and by integrating by parts (in time) the first term in the left hand side of (3.84), we obtain

$$\begin{aligned}
&\int_0^T\int_\Omega \frac{\partial\mathbf{v}}{\partial t}\cdot\widetilde{\mathbf{v}}\varphi(t) + \nu\int_0^T\int_\Omega \nabla\mathbf{v} : \nabla\widetilde{\mathbf{v}}\varphi(t) + \int_0^T\int_\Omega(\mathbf{v}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\varphi(t) + \int_0^T\int_\Omega(\mathbf{v}\cdot\nabla\mathbf{v}_s)\cdot\widetilde{\mathbf{v}}\varphi(t)\\
&\quad+\int_0^T\int_\Omega(\mathbf{v}_s\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\varphi(t) = -\frac{1}{2}\int_0^T\int_{\Gamma_s}(\mathbf{z}\cdot\widetilde{\mathbf{z}})(\mathbf{v}_s\cdot\mathbf{n})^-\varphi(t) - \frac{1}{2}\int_0^T\int_{\Gamma_s}(\mathbf{z}\cdot\widetilde{\mathbf{z}})(\mathbf{z}\cdot\mathbf{n})^-\varphi(t)\\
&\quad+\int_0^T \widetilde{\alpha}\mathcal{F}(\mathbf{v}, \alpha)\varphi(t),
\end{aligned}\tag{3.85}$$

for all $\widetilde{\mathbf{v}} = \widetilde{\mathbf{z}} + \widetilde{\alpha}\mathbf{w} \in W(Q)$ with $\widetilde{\mathbf{z}} \in \mathbf{Z}(\Omega)$ and $\widetilde{\alpha} \in \mathbb{R}$. Consequently, we obtain in the distribution sense on $]0,T[$

$$\int_\Omega \frac{\partial \mathbf{v}}{\partial t}\cdot\widetilde{\mathbf{v}} + \nu\int_\Omega \nabla\mathbf{v} : \nabla\widetilde{\mathbf{v}} + \int_\Omega (\mathbf{v}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}} + \int_\Omega (\mathbf{v}\cdot\nabla\mathbf{v}_s)\cdot\widetilde{\mathbf{v}} + \int_\Omega (\mathbf{v}_s\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}$$
$$= -\frac{1}{2}\int_{\Gamma_s}(\mathbf{z}\cdot\widetilde{\mathbf{z}})(\mathbf{v}_s\cdot\mathbf{n})^- - \frac{1}{2}\int_{\Gamma_s}(\mathbf{z}\cdot\widetilde{\mathbf{z}})(\mathbf{z}\cdot\mathbf{n})^- + \widetilde{\alpha}\mathcal{F}(\mathbf{v},\alpha) \ \text{ in } \mathcal{D}'(0,T) \tag{3.86}$$

for all $\widetilde{\mathbf{v}} = \widetilde{\mathbf{z}} + \widetilde{\alpha}\mathbf{w}$ and (3.49) is satisfied.

It now remains to retrieve the stabilized problem (3.9). Note that (3.9-b)-(3.9-d) are in $W(Q)$, and hence the three conditions are not examined in the following.

4.6 Retrieving the stabilized problem

First, we recall a result obtained in [31].

Lemma 4.7. *Let* $\mathbf{f} \in \mathcal{D}'(]0,T[;\Omega)$ *such that* $\langle \mathbf{f}, \widetilde{\mathbf{v}}\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = 0$ *for all* $\widetilde{\mathbf{v}} \in \mathcal{V}(\Omega)$. *Then there exists* $q \in \mathcal{D}'(]0,T[;L^2(\Omega))$ *such that* $\mathbf{f} = \nabla q$.

Lemma (4.7) is employed to prove the following result.

Lemma 4.8. *There exists* $p \in \mathcal{D}'(]0,T[;L^2(\Omega))$ *such that* $(\mathbf{v},p)$ *satisfies (3.9-a) in the distribution sense.*

Proof. For $\widetilde{\alpha} = 0$ and $\widetilde{\mathbf{z}} \in \mathcal{V}(\Omega)$, we have $\widetilde{\mathbf{v}} \in \mathcal{V}(\Omega)$. The particular choice $\widetilde{\mathbf{v}} \in \mathcal{V}(\Omega)$ in (3.85) leads to

$$\int_\Omega \frac{\partial \mathbf{v}}{\partial t}\cdot\widetilde{\mathbf{v}} + \nu\int_\Omega \nabla\mathbf{v} : \nabla\widetilde{\mathbf{v}} + \int_\Omega (\mathbf{v}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}} + \int_\Omega (\mathbf{v}\cdot\nabla\mathbf{v}_s)\cdot\widetilde{\mathbf{v}}$$
$$+ \int_\Omega (\mathbf{v}_s\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}} = 0, \ \text{ in } \mathcal{D}'(0,T). \tag{3.87}$$

By letting

$$\mathbf{f} = \frac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}_s + (\mathbf{v}_s\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v},$$

and using (3.87), we obtain $\mathbf{f} \in \mathcal{D}'(]0,T[;\Omega)$ and $\langle \mathbf{f}, \widetilde{\mathbf{v}}\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = 0$ for all $\widetilde{\mathbf{v}} \in \mathcal{V}(\Omega)$. Finally, using Lemma 4.7, there exists $p \in \mathcal{D}'(]0,T[;\, L^2(\Omega))$ such that $\mathbf{f} = -\nabla p$. □

We now prove that $(\mathbf{v},p)$ satisfies (3.9-e) and (3.9-g). Let us first define the space

$$E(\Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \mathbf{div}\ \mathbf{u} \in L^2(\Omega)\},$$

and recall the following Lemma obtained in [31, Chap I, Theorem 1.2].

Lemma 4.9. *Let Ω be an open bounded set of class $\mathcal{C}^2$. Then there exists a linear continuous operator $\gamma_n \in \mathcal{L}(E(\Omega), H^{-1/2}(\Gamma))$ such that $\gamma_n \mathbf{u}$ is the restriction of $\mathbf{u} \cdot \mathbf{n}$ to Γ, for every $\mathbf{u} \in \mathcal{D}(\overline{\Omega})$. The following generalized Stokes formula is true for all $\mathbf{u} \in E(\Omega)$ and $\mathbf{w} \in \mathbf{H}^1(\Omega)$,*

$$(\mathbf{u}, \nabla \mathbf{w}) + (div\ \mathbf{u}, \mathbf{w}) = \langle \gamma_n \mathbf{u}, \gamma_0 \mathbf{w} \rangle, \tag{3.88}$$

where $\gamma_0 \in \mathcal{L}(\mathbf{H}^1(\Omega), \mathbf{L}^2(\Gamma))$ is the trace operator.

By writing (3.9-a) in the form

$$\frac{\partial \mathbf{v}}{\partial t} + \mathrm{div}(-\nu \nabla \mathbf{v} + Ip) + (\mathbf{v} \cdot \nabla)\mathbf{v}_s + (\mathbf{v}_s \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = 0 \ \text{ in } Q_T,$$

and using Lemma 4.9, for all $(\widetilde{\mathbf{v}}, \widetilde{\alpha}) \in W(Q)$, we obtain

$$\int_\Omega \frac{\partial \mathbf{v}}{\partial t} \cdot \widetilde{\mathbf{v}} + \int_\Omega (\nu \nabla \mathbf{v} - Ip) : \nabla \widetilde{\mathbf{v}} + \langle (-\nu \nabla \mathbf{v} + Ip) \cdot \mathbf{n}, \widetilde{\mathbf{v}} \rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)}$$
$$+ \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}} + \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}_s) \cdot \widetilde{\mathbf{v}} + \int_\Omega (\mathbf{v}_s \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}} = 0.$$

Since $(\widetilde{\mathbf{v}}, \widetilde{\alpha}) \in W(Q)$, we have $pI : \nabla \widetilde{\mathbf{v}} = p \nabla \cdot \widetilde{\mathbf{v}} = 0$ and

$$\langle (-\nu \nabla \mathbf{v} + Ip) \cdot \mathbf{n}, \widetilde{\mathbf{v}} \rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} = -\widetilde{\alpha} \int_{\Gamma_e} [\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}] \cdot \mathbf{g} - \int_{\Gamma_s} [\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}] \cdot \widetilde{\mathbf{v}}.$$

Consequently,

$$\int_\Omega \frac{\partial \mathbf{v}}{\partial t} \cdot \widetilde{\mathbf{v}} + \nu \int_\Omega \nabla \mathbf{v} : \nabla \widetilde{\mathbf{v}} + \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}} + \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}_s) \cdot \widetilde{\mathbf{v}} + \int_\Omega (\mathbf{v}_s \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}}$$
$$= \widetilde{\alpha} \int_{\Gamma_e} [\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}] \cdot \mathbf{g} + \int_{\Gamma_s} [\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}] \cdot \widetilde{\mathbf{v}}. \tag{3.89}$$

Particularly, for $\widetilde{\alpha} = 0$, namely for any test function $\widetilde{\mathbf{v}}$ satisfying $\widetilde{\mathbf{v}} = 0$ on $\Gamma_l \cup \Gamma_e$, we have

$$\int_\Omega \frac{\partial \mathbf{v}}{\partial t} \cdot \widetilde{\mathbf{v}} + \nu \int_\Omega \nabla \mathbf{v} : \nabla \widetilde{\mathbf{v}} + \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}} + \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{v}_s) \cdot \widetilde{\mathbf{v}} + \int_\Omega (\mathbf{v}_s \cdot \nabla \mathbf{v}) \cdot \widetilde{\mathbf{v}}$$
$$= \int_{\Gamma_s} [\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}] \cdot \widetilde{\mathbf{v}}. \tag{3.90}$$

By comparing (3.86) with $\widetilde{\alpha} = 0$ and (3.90), we deduce

$$\int_{\Gamma_s} [\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}] \cdot \widetilde{\mathbf{v}} = -\frac{1}{2} \int_{\Gamma_s} (\mathbf{z} \cdot \widetilde{\mathbf{v}})(\mathbf{v}_s \cdot \mathbf{n})^- - \frac{1}{2} \int_{\Gamma_s} (\mathbf{z} \cdot \widetilde{\mathbf{v}})(\mathbf{z} \cdot \mathbf{n})^-, \ \forall \widetilde{\mathbf{v}} \in \mathbf{Z}(\Omega), \tag{3.91}$$

and hence

$$\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n} = -\frac{1}{2}\mathbf{z}\left((\mathbf{v}_s\cdot\mathbf{n})^- + (\mathbf{z}\cdot\mathbf{n})^-\right) \text{ on } \Sigma_s. \tag{3.92}$$

Finally, by inserting (3.92) into (3.89) and comparing with (3.86), we obtain

$$\int_{\Gamma_e}[\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}]\cdot\mathbf{g} = \mathcal{F}(\mathbf{v},\alpha). \tag{3.93}$$

According to (3.92) and (3.93), we retrieve (3.9-e) and (3.9-g), respectively.

In order to verify that the initial condition belongs to $\mathbf{H}(\Omega)$, we multiply (3.9-a) by $\widetilde{\mathbf{v}}\varphi$ with $\varphi(T)=0$ and integrate with respect to time and space

$$\begin{aligned}
&-\int_0^T\int_\Omega \mathbf{v}\cdot\widetilde{\mathbf{v}}\varphi'(t) + \nu\int_0^T\int_\Omega \nabla\mathbf{v}:\nabla\widetilde{\mathbf{v}}\varphi(t) + \int_0^T\int_\Omega(\mathbf{v}\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\varphi(t)\\
&\quad+\int_0^T\int_\Omega(\mathbf{v}\cdot\nabla\mathbf{v}_s)\cdot\widetilde{\mathbf{v}}\varphi(t) + \int_0^T\int_\Omega(\mathbf{v}_s\cdot\nabla\mathbf{v})\cdot\widetilde{\mathbf{v}}\varphi(t) - \int_\Omega \mathbf{v}(0)\widetilde{\mathbf{v}}\varphi(0)\\
&\quad\quad= -\frac{1}{2}\int_0^T\int_{\Gamma_s}(\mathbf{z}\cdot\widetilde{\mathbf{v}})(\mathbf{v}_s\cdot\mathbf{n})^-\varphi(t) - \frac{1}{2}\int_0^T\int_{\Gamma_s}(\mathbf{z}\cdot\widetilde{\mathbf{v}})(\mathbf{z}\cdot\mathbf{n})^-\varphi(t)\\
&\quad\quad\quad+\int_0^T\widetilde{\alpha}\mathcal{F}(\mathbf{v},\alpha)\varphi(t).
\end{aligned}\tag{3.94}$$

By comparing (3.84) and (3.94), we obtain $\int_\Omega(\mathbf{v}(0)-\mathbf{v}_0)\cdot\widetilde{\mathbf{v}}\varphi(0) = 0$, and choosing φ such that $\varphi(0)=1$, leads to

$$\int_\Omega(\mathbf{v}(0)-\mathbf{v}_0)\cdot\widetilde{\mathbf{v}} = 0, \quad \forall(\widetilde{\mathbf{v}},\widetilde{\alpha})\in W(Q).$$

Hence, as in chapter 1, $\mathbf{v}(0)=\mathbf{v}_0$ in $\mathcal{E}(Q)$ which is defined in (1.75).

5 Concluding remarks

In this work the exponential stabilization of the two dimensional Navier-Stokes equations in a bounded domain with mixed boundary conditions is studied around a given steady-state flow, using a boundary feedback control. In order to determine a feedback law, an extended system coupling the non stationary system (3.4), with equation (3.8) satisfied by the control on the domain boundary is considered.

Neumann and Dirichlet boundary conditions are imposed and the velocity boundary control $\mathbf{u}_b = \alpha\mathbf{g}$ is satisfied on the Dirichlet part. The velocity profile g satisfies (3.52) and (3.17) and the proportionality coefficient α, an unknown of the problem, measures the velocity flux magnitude at the interface. Note that the size of the initial velocity $\mathbf{v}_0(\mathbf{x})$ is arbitrary and does not need to be bounded. A Galerkin method is employed, and α is determined such that the boundary control $\mathbf{u}_b$ is satisfied on a part of the Dirichlet boundary, and the stabilizing boundary control is built. The resulting feedback control is proven to be globally exponentially stabilizing the steady states of the Navier-Stokes equations. This feedback control is shown to guarantee global stability in the L^2-norm.

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Chapitre 4

Numerical feedback stabilization of the Navier-Stokes equations using the characteristic-Galerkin method

Abstract

In this work we study the numerical feedback stabilization of the two and three-dimensional Navier-Stokes equations in a bounded domain Ω, around a given steady-state flow, by means of a boundary control. In order to determine a feedback law, we consider an extended system coupling the Navier-Stokes equations with an equation satisfied by the control on the domain boundary. While most traditional approaches apply a feedback controller via an algebraic Bernouilli equation (ABE) or a model reduction, a characteristic-Galerkin method is proposed instead in this study. The characteristic-Galerkin method permits to construct a stabilizing boundary control by solving a polynomial equation of degree one or two. Further, by using energy a priori estimation techniques, the exponential decay is obtained. The numerical relevance of this approach is illustrated by stabilizing the two-dimensional flow problem, around a circular obstacle.

Keywords : Navier-Stokes system, feedback control, boundary stabilization, Galerkin method.

1 Introduction

Let Ω be a bounded and connected domain in $\mathbb{R}^d,\ d = 2, 3$, with a boundary Γ of class C^2, and composed of three connected components Γ_b, Γ_l and Γ_s such that $\Gamma = \Gamma_b \cup \Gamma_l \cup \Gamma_s$. In particular, the boundary Γ_b is the part of Γ, where a Dirichlet boundary control in feedback form has to be determined. The usual function spaces $L^2(\Omega),\ H^s(\Omega),\ H_0^s(\Omega)(s > 0)$ are used and we let $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{H}^s(\Omega) = (H^s(\Omega))^d$ and $\mathbf{H}_0^s(\Omega) = (H_0^s(\Omega))^d$. The same conventions are used for spaces of traces $L^2(\Gamma)$ and $H^s(\Gamma)$. Finally, we denote by $\langle \cdot \mid \cdot \rangle$ and $\| \cdot \| = \| \cdot \|_{\mathbf{L}^2(\Omega)}$, the scalar product and norm in $\mathbf{L}^2(\Omega)$, respectively.

Consider a stationary motion of an incompressible fluid described by the velocity and pressure couple $(\mathbf{v}_s, q_s)$ solution to the stationary Navier-Stokes equations

$$\begin{cases} -\nu\Delta\mathbf{v}_s + (\mathbf{v}_s.\nabla)\mathbf{v}_s + \nabla q_s = \mathbf{f}_s & \text{in } \Omega, \\ \nabla\cdot\mathbf{v}_s = 0 & \text{in } \Omega, \\ \mathbf{v}_s = 0 & \text{on } \Gamma_l, \\ \mathbf{v}_s = \mathbf{v}_b & \text{on } \Gamma_b, \\ \nu\nabla\mathbf{v}_s\cdot\mathbf{n} - q_s\mathbf{n} = 0 & \text{on } \Gamma_s, \end{cases} \tag{4.1}$$

where $\mathbf{n}$ is the unit outer normal vector to Γ, $\mathbf{f}_s$ represents body forces acting on the fluid and $\mathbf{v}_b$ denotes a specified boundary velocity. Further, $R_e = \frac{U_0 L_0}{\nu}$ is the Reynolds number, with ν, L_0 and U_0 being the kinematic viscosity, characteristic length and characteristic velocity, respectively.

For $T > 0$ a fixed real number, we let $Q = [0,T)\times\Omega$, $\Sigma_b = [0,T)\times\Gamma_b$, $\Sigma_l = [0,T)\times\Gamma_l$ and $\Sigma_s = [0,T)\times\Gamma_s$. Further, $\boldsymbol{\psi}(t,\mathbf{x})$ and $q(t,\mathbf{x})$ denote the velocity and pressure fields, respectively. The initial boundary value problem associated with the non-stationary incompressible Navier-Stokes system is then given by

$$\begin{cases} \dfrac{\partial\boldsymbol{\psi}}{\partial t} - \nu\Delta\boldsymbol{\psi} + (\boldsymbol{\psi}\cdot\nabla)\boldsymbol{\psi} + \nabla q = \mathbf{f}_s & \text{in } Q, \\ \nabla\cdot\boldsymbol{\psi} = 0 & \text{in } Q, \\ \boldsymbol{\psi} = 0 & \text{on } \Sigma_l, \\ \boldsymbol{\psi} = \mathbf{u}_b + \mathbf{v}_b & \text{on } \Sigma_b, \\ \nu\nabla\boldsymbol{\psi}\cdot\mathbf{n} - q\mathbf{n} = 0 & \text{on } \Sigma_s, \\ \boldsymbol{\psi}(0,\mathbf{x}) = \mathbf{v}_s + \mathbf{v}_0 & \text{in } \ \Omega. \end{cases} \tag{4.2}$$

The function $\mathbf{u}_b$ is the control input and the function $\mathbf{v}_0$ can be viewed as a perturbation of the initial state. By substituting $(\boldsymbol{\psi}, q) = (\mathbf{v}_s + \mathbf{v}, q_s + p)$ in (4.2), we obtain

$$\begin{cases} (a) \quad \dfrac{\partial\mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}_s + (\mathbf{v}_s\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0 & \text{in } Q, \\ (b) \quad \nabla\cdot\ \mathbf{v} = 0 & \text{in } Q, \\ (c) \quad \mathbf{v} = 0 & \text{on } \Sigma_l, \\ (d) \quad \mathbf{v} = \mathbf{u}_b, & \text{on } \Sigma_b, \\ (e) \quad \nu\nabla\mathbf{v}\cdot\mathbf{n} - p\mathbf{n} = 0 & \text{on } \Sigma_s, \\ (f) \quad \mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{in } \ \Omega. \end{cases} \tag{4.3}$$

The goal of this study is to determine a control law $\mathcal{M}$ on $\mathbb{R}\times\Gamma_b$ in the form of a state feedback law $\mathbf{u}_b = \mathcal{M}(\mathbf{v})$ such that

$$\|\mathbf{v}(t)\| \leq C\|\mathbf{v}_0\|e^{-\mu t}, \quad \forall t > 0, \tag{4.4}$$

with a prescribed rate of decrease $\mu > 0$.

The theoretical setting of the stabilization procedure, for the non-stationary incompressible Navier-Stokes equations using a feedback control, has been studied by a number of authors, e.g. A.V. Fursikov [19, 20], V. Barbu et al. [5, 9, 10, 11, 12], J.-P. Raymond et al. [33, 34, 35] and M. Badra et al. [2, 3, 4]. In these papers, the linear feedback law $\mathcal{M}$ on $\mathbb{R} \times \Gamma_b$ is first determined by solving a linear control problem for the linearized system of equations (for example the Oseen system) and then this linear feedback is used in order to stabilize the original non linear system. However, the development of fast computational algorithms for feedback control design of fluid dynamic systems is hindered by a few intrinsic difficulties. Indeed, in [3, 4, 8, 9, 10, 31, 34, 35], the feedback control laws are determined by solving a Riccati equation in a space of infinite dimension. In such a case, an optimal control problem has to be solved, involving the minimization of an objective functional. In practice, the control is calculated through an approximation via the solution of an algebraic Riccati equation (ARE), which may be computationally expensive. The use of finite-dimensional controllers may be more appropriate to stabilize the Navier-Stokes equations. Such an approach is performed in [2, 5, 8, 9, 33] without numerical experiments, and in [1, 32] with a few numerical illustrations.

In [32], the author consider the optimal boundary feedback stabilization of fluid flows governed by the Navier-Stokes equations using model reduction. The model reduction is carried out using a combination of proper orthogonal decomposition (POD) and Galerkin projection. The resulting reduced-order model is employed in the optimal linear quadratic regulator (LQR) design to derive a feedback control. The feedback control is then used in the nonlinear Navier-Stokes equations to stabilize the system. However, the problem of rigorously proving that the finite-dimensional reduced-order controllers proposed in [32] is able to stabilize the infinite dimensional model is not addressed in [32] and this is still an outstanding problem.

In [1], the authors obtain the feedback operator $\mathcal{M}$ from the solution of the algebraic Bernoulli equation (ABE) associated with the penalized linearized Navier-Stokes equations around an unstable stationary solution. The operator $\mathcal{M}$ is then used to locally stabilize the original nonlinear equations. As mentioned in [1], if k is the rank of $\mathcal{M}$, the ABE is particularly relevant when k is small, compared with the size of the problem. This is the case for the Navier-Stokes equations at low Reynolds regimes $R_e \leq 200$, that are considered in [1].

A linear feedback law is first determined by solving a linear control problem in all the papers cited above, and this linear feedback is then used in order to stabilize the original non linear system. Such a procedure leads to choose the initial velocity small enough and it usually requires to search for the control $\mathbf{u}_b$ and the initial condition in sufficiently regular spaces. This is why another approach is proposed in [29], where an extended system is considered with an additional equation satisfied by the control on the domain boundary, and the boundary feedback control is constructed via a Galerkin method. The boundary control $\mathbf{u}_b$ in (4.3) is rewritten on the form

$$\mathbf{u}_b = \alpha(t)\mathbf{g}(\mathbf{x}) \text{ on } \Sigma_b, \tag{4.5}$$

where $\mathbf{g} \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$ is assumed to verify $\mathbf{g} \cdot \mathbf{n} \neq 0$ on Γ_b and $\int_{\Gamma_b} \mathbf{g} \cdot \mathbf{n} = 0$. The quantity $\alpha(t)$ is a priori unknown. In order to stabilize the Navier-Stokes system, with $\mathbf{u}_b = \alpha(t)\mathbf{g}(\mathbf{x})$ on Σ_b, by employing energy a priori estimation techniques, the quantity $\alpha(t)$ is found to satisfy the relation

$$\int_{\Gamma_b} [\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}] \cdot \mathbf{g} = \mathcal{F}(\mathbf{v}, \alpha), \tag{4.6}$$

where $\mathcal{F}$ is a second order polynomial with to respect to α. The quantity $\alpha(t)$ depends nonlinearly on $\mathbf{v}$ in (4.6), and hence $\alpha(t)$ satisfies a nonlinear feedback law. However in practice, because (4.5) and (4.6) are defined at the same boundary Γ_b, the numerical methods for discretizing (4.5) and (4.6) cannot be easily implemented. The goal of this study is to develop a practical computational algorithm easy to implement. In this respect, the characteristic-Galerkin method is employed to search for $(\mathbf{v}, p)$, solution of (4.3). Let us denote by $(\mathbf{v}^n, p^n)$ the approximations of the velocity and pressure $(\mathbf{v}, p)$ at time t^n, and α_n the approximation of the control α at time t^n. After the time discretization is performed, a linear system is obtained and $(\mathbf{v}^n, p^n)$ is decomposed as

$$\begin{cases} \mathbf{v}^n &= \widetilde{\mathbf{w}}^n + \alpha_n \mathbf{w} \\ p^n &= \widetilde{q}^n + \alpha_n q, \end{cases} \tag{4.7}$$

where

(i) $(\mathbf{w}, q)$ does not depend on time and $\mathbf{w}$ satisfies $\mathbf{w} = 0$ on Γ_l and $\mathbf{w} = \mathbf{g}$ on Γ_b.

(ii) $(\widetilde{\mathbf{w}}^n, \widetilde{q}^n)$ depends on time but does not depend on α_n and $\widetilde{\mathbf{w}}^n$ satisfies $\widetilde{\mathbf{w}}^n = 0$ on $\Gamma_b \cup \Gamma_l$.

(iii) The control α_n is searched such that $\alpha_n = \mathcal{M}(\mathbf{v}^n)$, where $\mathcal{M}$ is specified later in (4.37) or (4.48).

The paper is organized as follows. In section 2, the notations and mathematical preliminaries are given. The feedback law is defined in Section 3 thanks to technics developed in [17], which are not related specifically to a stabilization problem, and the characteristic-Galerkin method. Finally, we illustrate numerically the effectiveness of the method by stabilizing the Navier-Stokes equations around a circular obstacle.

2 Notations and Preliminaries

2.1 Notations

Spaces of free divergence functions are introduced :

$$\mathbf{V}(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) \ : \ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, \ \mathbf{u} = 0 \text{ on } \Gamma_l\}, \tag{4.8}$$

$$\mathbf{V}_0(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) \ : \ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, \ \mathbf{u} = 0 \text{ on } \Gamma_b \cup \Gamma_l\}, \tag{4.9}$$

$$\mathbf{H}(\Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega) \ : \nabla \cdot \mathbf{u} = 0, \ \ \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma_l\}. \tag{4.10}$$

Let us denote by $V^{\frac{1}{2}}(\Gamma_b)$ the space of functions whose extension by zero over Γ belong to $\mathbf{H}^{\frac{1}{2}}(\Gamma)$. For $\mathbf{g} \in V^{\frac{1}{2}}(\Gamma_b)$ with $\mathbf{g} \neq 0$, we define the space of solution

$$W(Q) = \{(\mathbf{v}, \alpha) \in \mathbf{V}(\Omega) \times \mathbb{R}, \text{ s.t. } \mathbf{v} = \alpha \mathbf{g} \text{ on } \Gamma_b\}.$$

In order to define a weak form of the Navier-Stokes equations, we introduce the continuous bilinear forms

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v}, \quad \forall \mathbf{u}, \, \mathbf{v} \in \mathbf{H}^1(\Omega), \\ b(\mathbf{v}, q) &= \int_\Omega q \, \nabla \mathbf{v}, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \ \forall q \in L^2(\Omega). \end{aligned}$$

2.2 Preliminaries

For an initial data $\mathbf{v}_0$ belonging to an appropriate functional space, we search for the numerical solution of the stabilization problem (4.3) by using the characteristic-Galerkin method. Let $\mathbf{v}$ be the velocity field of the fluid, and denote by $X(\tau; t, \mathbf{x})$ and $Y(\tau; t, \mathbf{x})$ the solutions of the following ordinary differential equation in τ

$$\begin{cases} (a) \quad \dfrac{dX}{d\tau} = \mathbf{v}(\tau, X(\tau; t, \mathbf{x})) & \text{if } X(\tau; t, \mathbf{x}) \in \Omega, \\ \qquad\quad\ = 0 & \text{otherwise,} \\ (b) \quad X(t; t, \mathbf{x}) = \mathbf{x}, & \end{cases} \tag{4.11}$$

$$\begin{cases} (a) & \dfrac{dY}{d\tau} = \mathbf{v}(\tau, Y(\tau; t, \mathbf{x})) + 2\,\mathbf{v}_s(Y(\tau; t, \mathbf{x})) & \text{if } Y(\tau; t, \mathbf{x}) \in \Omega, \\ & \quad = 0 & \text{otherwise}, \\ (b) & Y(t; t, \mathbf{x}) = \mathbf{x}, & \end{cases} \tag{4.12}$$

where $X(\cdot\,;\,t, \mathbf{x})$ and $Y(\cdot\,;\,t, \mathbf{x})$ are the particle path that passes at $\mathbf{x} = (x_1, x_2)$ at time t. Let $\frac{D}{Dt}$ denotes the material derivative (also called Lagrangian derivative) of the velocity field. We have

$$\begin{aligned} \frac{1}{2}\frac{D\mathbf{v}}{Dt}(t, \mathbf{x}) &= \frac{1}{2}\left[\frac{D\mathbf{v}}{D\tau}(\tau, Y(\tau; t, \mathbf{x}))\right]_{\tau=t} \\ &= \frac{1}{2}\left[\frac{dY}{d\tau}\nabla\mathbf{v}(\tau, Y(\tau; t, \mathbf{x})) + \frac{\partial\mathbf{v}}{\partial\tau}(\tau, Y(\tau; t, \mathbf{x}))\right]_{\tau=t}. \end{aligned} \tag{4.13}$$

Using (4.12) and (4.13), we deduce

$$\frac{1}{2}\frac{D\mathbf{v}}{Dt}(t, \mathbf{x}) = \frac{1}{2}\frac{\partial\mathbf{v}}{\partial t} + \frac{1}{2}(\mathbf{v}\cdot\nabla)\mathbf{v} + (\mathbf{v}_s\cdot\nabla)\mathbf{v}. \tag{4.14}$$

Similarly, we obtain

$$\begin{aligned} \frac{1}{2}\frac{D(\mathbf{v} + 2\,\mathbf{v}_s)}{Dt}(t, \mathbf{x}) &= \frac{1}{2}\left[\frac{D(\mathbf{v} + 2\,\mathbf{v}_s)}{D\tau}(\tau, X(\tau; t, \mathbf{x}))\right]_{\tau=t} \\ &= \frac{1}{2}\left[\frac{dX}{d\tau}\nabla(\mathbf{v} + 2\,\mathbf{v}_s)(\tau, X(\tau; t, \mathbf{x})) + \frac{\partial\mathbf{v}}{\partial\tau}(\tau, X(\tau; t, \mathbf{x}))\right]_{\tau=t}. \end{aligned} \tag{4.15}$$

Using (4.11) and (4.15), we deduce

$$\frac{1}{2}\frac{D(\mathbf{v} + 2\,\mathbf{v}_s)}{Dt} = \frac{1}{2}\frac{\partial\mathbf{v}}{\partial t} + \frac{1}{2}(\mathbf{v}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}_s. \tag{4.16}$$

Thus, summing (4.14) and (4.16), problem (4.3) can be rewritten as

$$\begin{cases} (a) & \dfrac{1}{2}\dfrac{D\mathbf{v}}{Dt} + \dfrac{1}{2}\dfrac{D(\mathbf{v} + 2\,\mathbf{v}_s)}{Dt} - \nu\Delta\mathbf{v} + \nabla p = 0 & \text{in } Q, \\ (b) & \nabla\cdot\;\mathbf{v} = 0 & \text{in } Q, \\ (c) & \mathbf{v} = 0 & \text{on } \Sigma_l, \\ (d) & \mathbf{v} = \alpha(t)\mathbf{g}(\mathbf{x}) & \text{on } \Sigma_b, \\ (e) & \nu\nabla\mathbf{v}\cdot\mathbf{n} - p\mathbf{n} = 0 & \text{on } \Sigma_s, \\ (f) & \mathbf{v}(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega. \end{cases} \tag{4.17}$$

3 Time discretization and Assumptions

3.1 Time discretization of stabilization problem

Let $t^0 = 0 < t^1 < t^2 < \cdots < t^N = T$ with $t^n - t^{n-1} = \Delta t = T/N$ denotes the time step. We propose a time discretization of the material derivative, e.g. by means of the backward Euler method. Using (4.13) and (4.15) respectively, we obtain

$$\frac{D\mathbf{v}}{Dt}(t^n,\mathbf{x}) \approx \frac{\mathbf{v}(t^n, Y(t^n;t^n,\mathbf{x})) - \mathbf{v}(t^{n-1}, Y(t^{n-1};t^n,\mathbf{x}))}{\Delta t}, \tag{4.18}$$

$$\frac{D\mathbf{u}}{Dt}(t^n,\mathbf{x}) \approx \frac{\mathbf{u}(t^n, X(t^n;t^n,\mathbf{x})) - \mathbf{u}(t^{n-1}, X(t^{n-1};t^n,\mathbf{x}))}{\Delta t}, \tag{4.19}$$

where $\mathbf{u} = \mathbf{v} + 2\,\mathbf{v}_s$. The characteristics foot $X(t^{n-1};t^n,\mathbf{x})$ and $Y(t^{n-1};t^n,\mathbf{x})$ are computed from (4.11) and (4.12), respectively using the following linear discrete interpolation :

$$X(t^{n-1};t^n,\mathbf{x}) \approx \mathbf{x} - \mathbf{v}(t^n,\mathbf{x})\Delta t,$$

$$Y(t^{n-1};t^n,\mathbf{x}) \approx \mathbf{x} - \mathbf{u}(t^n,\mathbf{x})\Delta t.$$

Due to (4.11-b) and (4.12-b) , we have $X(t^n;t^n,\mathbf{x}) = Y(t^n;t^n,\mathbf{x}) = \mathbf{x}$ and hence (4.18) and (4.19) become respectively

$$\frac{D\mathbf{v}}{Dt}(t^n,\mathbf{x}) \approx \frac{\mathbf{v}(t^n,\mathbf{x}) - \mathbf{v}(t^{n-1}, Y(t^{n-1},\mathbf{x}))}{\Delta t}, \tag{4.20}$$

$$\frac{D\mathbf{u}}{Dt}(t^n,\mathbf{x}) \approx \frac{\mathbf{u}(t^n,\mathbf{x}) - \mathbf{u}(t^{n-1}, X(t^{n-1},\mathbf{x}))}{\Delta t}, \tag{4.21}$$

where $X(t^{n-1},\mathbf{x}) = X(t^{n-1};t^n,\mathbf{x})$ and $Y(t^{n-1},\mathbf{x}) = Y(t^{n-1};t^n,\mathbf{x})$. Setting $(\mathbf{v}^n, p^n) = (\mathbf{v}, p)(t^n,\mathbf{x})$, the approximations of the velocity and pressure at time t^n, the time discretization of the stabilization system (4.17) leads to

$$\begin{cases} (a) & \dfrac{\mathbf{v}^n}{\Delta t} - \nu\Delta\mathbf{v}^n + \nabla p^n = \dfrac{F^{n-1}}{\Delta t} & \text{in } \Omega, \\ (b) & \nabla\cdot\mathbf{v}^n = 0 & \text{in } \Omega, \\ (c) & \mathbf{v}^n = 0 & \text{on } \Gamma_l, \\ (d) & \mathbf{v}^n = \alpha_n \mathbf{g}(\mathbf{x}) & \text{on } \Gamma_b, \\ (e) & \nu\nabla\mathbf{v}^n\cdot\mathbf{n} - p^n\mathbf{n} = 0 & \text{on } \Sigma_s, \\ (f) & \mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{4.22}$$

with

$$F^{n-1} = \frac{1}{2}\big(\mathbf{v}^{n-1} \circ X^{n-1} + \mathbf{v}^{n-1} \circ Y^{n-1}\big) + \mathbf{v}_s \circ X^{n-1} - \mathbf{v}_s, \tag{4.23}$$

where $\mathbf{v} \circ Z$ denotes the function $\mathbf{x} \to \mathbf{v}[Z(\mathbf{x})]$.

For an initial data v_0 in an appropriate functional space, our goal is to find a feedback control α_n such that v^n, solution of the system (4.22), satisfies (4.4).

3.2 Controller building process

Since system (4.22) is linear, the solution $(\mathbf{v}^n, p^n)$ is decomposed as

$$\begin{cases} \mathbf{v}^n &= \widetilde{\mathbf{w}}^n + \alpha_n \mathbf{w}, \\ p^n &= \widetilde{q}^n + \alpha_n q, \end{cases} \tag{4.24}$$

where $(\mathbf{w}, q)$ does not depend on time, while the couple $(\widetilde{\mathbf{w}}^n, \widetilde{q}^n)$ represents correction terms which are calculated at each time step. The details of the controller building process is specified as follows :

(i) Firstly, we search for $(\mathbf{w}, q)$ such that

$$\begin{cases} (a) & \dfrac{\mathbf{w}}{\Delta t} - \nu\Delta\mathbf{w} + \nabla q = 0 & \text{in } \Omega, \\ (b) & \nabla \cdot \mathbf{w} = 0 & \text{in } \Omega, \\ (c) & \mathbf{w} = 0 & \text{on } \Gamma_l, \\ (d) & \mathbf{w} = \mathbf{g} & \text{on } \Gamma_b \\ (e) & \nu\nabla\mathbf{w}\cdot\mathbf{n} - q\mathbf{n} = 0 & \text{on } \Gamma_s. \end{cases} \tag{4.25}$$

(ii) Secondly, at each time step, we search for $(\widetilde{\mathbf{w}}^n, \widetilde{q}^n)$ such that

$$\begin{cases} (a) & \dfrac{\widetilde{\mathbf{w}}^n}{\Delta t} - \nu\Delta\widetilde{\mathbf{w}}^n + \nabla\widetilde{q}^n = \dfrac{F^{n-1}}{\Delta t} & \text{in } \Omega, \\ (b) & \nabla \cdot \widetilde{\mathbf{w}}^n = 0 & \text{in } \Omega, \\ (c) & \widetilde{\mathbf{w}}^n = 0 & \text{on } \Gamma_l \cup \Gamma_b \\ (d) & \nu\nabla\widetilde{\mathbf{w}}^n\cdot\mathbf{n} - \widetilde{q}^n\mathbf{n} = 0 & \text{on } \Gamma_s. \end{cases} \tag{4.26}$$

(iii) Finally, in order to stabilize (4.22) with $\mathbf{v}^n = \alpha_n \mathbf{g}(\mathbf{x})$ on Γ_b, by employing energy a priori estimation techniques, the quantity α_n needs to satisfy the relation

$$\int_{\Gamma_b} [\nu\nabla\mathbf{v}^n\cdot\mathbf{n} - p^n\mathbf{n}]\cdot\mathbf{g} = -\lambda\alpha_n, \quad \lambda > 0. \tag{4.27}$$

Such a procedure relies on technics previously introduced in [17], but it is worth to note that the work performed in [17] is not related to a stabilization problem. To show the stability result, we need the following assumptions.

3.3 Assumptions and main result

Firstly, for all $n \in \mathbb{N}$, we assume that

$$X^n(\mathbf{x}) = \mathbf{x} - \mathbf{v}^n(\mathbf{x})\,\Delta t \in \Omega, \tag{4.28}$$

$$Y^n(\mathbf{x}) = \mathbf{x} - \mathbf{u}^n(\mathbf{x})\,\Delta t \in \Omega. \tag{4.29}$$

Note that the classical spatial approximation of the characteristic curves (4.28)-(4.29) has been used in a number of papers, e.g. in [18, 25, 30]. Such assumptions mean that the foots of the characteristic curves are not allowed to lie outside the domain boundary. In practice, the foots of the characteristic curves may lie outside the domain boundary due to (small) space and time truncation errors of the numerical method, and in such a case they are projected orthogonally on the domain boundary.

Secondly, by using the Taylor's theorem for multivariate functions, we obtain

$$\mathbf{v}_s\big(\mathbf{x} - \mathbf{v}^n(\mathbf{x})\,\Delta t\big) = \mathbf{v}_s(\mathbf{x}) - \Delta t \nabla \mathbf{v}_s(\mathbf{x}) \cdot \mathbf{v}^n(\mathbf{x}) + O(\Delta t^2).$$

Hence, by neglecting the second order term, we obtain the following assumption

$$\mathbf{v}_s\big(\mathbf{x} - \mathbf{v}^n(\mathbf{x})\,\Delta t\big) = \mathbf{v}_s(\mathbf{x}) - \Delta t \nabla \mathbf{v}_s(\mathbf{x}) \cdot \mathbf{v}^n(\mathbf{x}). \tag{4.30}$$

Lemma 3.1. *Under the assumptions (4.28)-(4.29) and (4.30), the following assumption holds*

$$\|F^n\| \leq \big(1 + \Delta t \|\nabla \mathbf{v}_s\|\big)\|\mathbf{v}^n\|. \tag{4.31}$$

Proof. According to (4.23) and using (4.30), we obtain

$$\|F^n\| \leq \frac{1}{2}\|\mathbf{v}^n \circ X^n\| + \frac{1}{2}\|\mathbf{v}^n \circ Y^n\| + \Delta t \|\nabla \mathbf{v}_s\| \|\mathbf{v}^n\|. \tag{4.32}$$

Let $Z^n = X^n$ or Y^n and let J^n be the Jacobian matrix of the transformation $\mathbf{y} = Z^n(\mathbf{x})$, we obtain

$$\|\mathbf{v}^n \circ Z^n\|^2 = \int_\Omega (\mathbf{v}^n[Z^n(\mathbf{x})])^2 d\mathbf{x} = \int_{Z^n(\Omega)} (\mathbf{v}^n(\mathbf{y}))^2 (\det J^n)^{-1} d\mathbf{y}.$$

By definition, we have from [13]

$$J(\tau; t, \mathbf{x}) = -\int_\tau^t \nabla \cdot \mathbf{v}(\tau, Z(\tau; t, \mathbf{x})) d\tau + 1, \tag{4.33}$$

and since $\nabla \cdot \mathbf{v}^n = 0$, using (4.33) yields det $J^n = 1$. Further, $Z^n(\Omega) \subset \Omega$, and hence

$$\|\mathbf{v}^n \circ Z^n\|^2 = \int_{Z^n(\Omega)} (\mathbf{v}^n(\mathbf{y}))^2 (\text{ det } J)^{-1} d\mathbf{y} \leq \int_{\Omega} (\mathbf{v}^n(\mathbf{x}))^2 d\mathbf{x} = \|\mathbf{v}^n\|^2. \tag{4.34}$$

Inserting (4.34) in (4.32), we deduce (4.31). □

In the following we attempt to find a boundary feedback control α_n, with a control law similar to that employed in the first three chapters of this thesis i.e

$$\begin{cases} \mathbf{v}^n = \alpha_n \mathbf{g} & \text{in } \Gamma_b, \\ \displaystyle\int_{\Gamma_b} [\nu \nabla \mathbf{v}^n \cdot \mathbf{n} - p^n \mathbf{n}] \cdot \mathbf{g} = f(\alpha_n), \end{cases} \tag{4.35}$$

and this leads to the following proposition.

Proposition 3.1. *Let* $\mathbf{v}_0 \in \mathbf{H}(\Omega)$, $\mathbf{g} \in V^{\frac{1}{2}}(\Gamma_b)$ *with* $\mathbf{g} \neq 0$ *on* Γ_b *and* $\mathbf{v}_s$ *such that*

$$\|\nabla \mathbf{v}_s\| \leq \frac{1}{\Delta t} \left(\sqrt{1 + \frac{2\,\nu \Delta\, t}{C_p^2}} - 1 \right), \tag{4.36}$$

where C_p *is the Poincaré constant. Under the assumptions (4.28)-(4.29) and (4.30), there exists a boundary feedback control* α_n *on* Γ_b *solution of*

$$\int_{\Gamma_b} [\nu \nabla \mathbf{v}^n \cdot \mathbf{n} - p^n \mathbf{n}] \cdot \mathbf{g} = -\lambda \alpha_n, \quad \lambda > 0 \tag{4.37}$$

such that system (4.22) with $(\mathbf{v}^n, p^n)$ *writen as in (4.24) is exponentially stable. i.e. there exists* $\mu > 0$ *such that* $\mathbf{v}^n$ *satisfies*

$$\|\mathbf{v}^n\| \quad \leq \quad \|\mathbf{v}_0\| \exp\left(-\mu t^n\right). \tag{4.38}$$

Proof. We define

$$B(\mathbf{z}, \pi) = \int_{\Gamma_b} [\nu \nabla \mathbf{z} \cdot \mathbf{n} - \pi \mathbf{n}] \cdot \mathbf{g},$$

and using (4.24) we obtain

$$B(\mathbf{v}^n, p^n) = \alpha_n B(\mathbf{w}, q) + B(\widetilde{\mathbf{w}}^n, \widetilde{q}^n).$$

Consequently, $\alpha_n = -\dfrac{B(\widetilde{\mathbf{w}}^n, \widetilde{q}^n)}{\lambda + B(\mathbf{w}, q)}$ satisfies (4.37).

The variational formulation of (4.22) is defined as

$$\frac{1}{\Delta t}\langle \mathbf{v}^n, \widetilde{\mathbf{v}}\rangle + \nu a(\mathbf{v}^n, \widetilde{\mathbf{v}}) = \frac{1}{\Delta t}\langle F^{n-1}, \widetilde{\mathbf{v}}\rangle + \widetilde{\alpha}\int_{\Gamma_b}[\nu\nabla \mathbf{v}^n\cdot\mathbf{n} - p^n\mathbf{n}]\cdot\mathbf{g}, \tag{4.39}$$

for all $(\widetilde{\mathbf{v}}, \widetilde{\alpha}) \in W(Q)$. Taking $\widetilde{\mathbf{v}} = \mathbf{v}^n$ in (4.39) and employing (4.23) yields

$$\frac{1}{2\,\Delta t}\|\mathbf{v}^n\|^2 + \nu\|\nabla\mathbf{v}^n\|^2 \leq \frac{1}{2\Delta t}\|F^{n-1}\|^2 + \alpha_n\int_{\Gamma_b}[\nu\nabla \mathbf{v}^n\cdot\mathbf{n} - p^n\mathbf{n}]\cdot\mathbf{g}. \tag{4.40}$$

Using (4.37) in (4.40), we obtain

$$\|\mathbf{v}^n\|^2 + 2\nu\Delta t\|\nabla\mathbf{v}^n\|^2 + 2\lambda\Delta t\alpha_n^2 \leq \|F^{n-1}\|^2. \tag{4.41}$$

By using Lemma 3.1 and Poincaré inequality in (4.41), we obtain

$$\sqrt{C_p^2 + 2\nu\Delta\,t}\|\mathbf{v}^n\| \leq C_p\big(1 + \Delta t\|\nabla\mathbf{v}_s\|\big)\|\mathbf{v}^{n-1}\|.$$

and hence, we deduce that

$$\|\mathbf{v}^n\| \leq \theta\|\mathbf{v}^{n-1}\| \;\text{ with }\; \theta = \frac{1 + \Delta t\,\|\nabla\mathbf{v}_s\|}{\sqrt{1 + \dfrac{2\nu\Delta\,t}{C_p^2}}}. \tag{4.42}$$

According to (4.36), $\theta < 1$ and recursively we obtain

$$\|\mathbf{v}^n\| = \|\mathbf{v}(t^n)\| \leq \theta^n\|\mathbf{v}_0\|. \tag{4.43}$$

To achieve the proof, we show that (4.43) implies (4.45). Taking $\mu = -\dfrac{ln(\theta)}{\Delta\,t} > 0$, and using (4.43) leads to

$$\begin{aligned}\|\mathbf{v}^n\| \leq \theta^n\|\mathbf{v}_0\| &= \exp\left(n\,ln(\theta)\right)\|\mathbf{v}_0\| = \exp\left(-\mu\,n\,\Delta\,t\right)\|\mathbf{v}_0\|,\\ &= \exp\left(-\mu\,t^n\right)\|\mathbf{v}_0\|,\end{aligned}$$

and hence, estimate (4.45) is obtained. □

For equilibrium states $\mathrm{v_s}$ corresponding to small Reynolds numbers, whatever the initial velocity, Proposition 3.1 may be employed, and an exponential decrease of the energy is obtained. However, it is difficult to find the appropriate interval for α_n in order to obtain an optimal decrease. This suggests to employ a more appropriate control law in order to find such an inteval for α_n, and it is the subject of the following proposition.

Proposition 3.2. *Under assumptions (4.28)-(4.30) and (4.36), the solution $\widetilde{\mathrm{w}}^n$ of (4.26)* *satisfies*

$$\|\widetilde{\mathbf{w}}^n\| \leq \theta \|\mathbf{v}^{n-1}\|, \tag{4.44}$$

where θ is defined in (4.42). Consequently, there exists a boundary feedback control α_n, solution of a polynomial of degree two such that system (4.22) with $(\mathbf{v}^n, p^n)$ writen as in (4.24) is exponentially stable. i.e. there exists $\mu > 0$ such that $\mathbf{v}^n$ satisfies

$$\|\mathbf{v}^n\| \quad \leq \quad \|\mathbf{v}_0\| \exp\left(-\mu t^n\right). \tag{4.45}$$

Proof. The varitional formulation of (4.26) is defined as

$$\frac{1}{\Delta\, t}\langle \widetilde{\mathbf{w}}^n, \widetilde{\mathbf{v}}\rangle + \nu a(\widetilde{\mathbf{w}}^n, \widetilde{\mathbf{v}}) = \frac{1}{\Delta\, t}\langle F^{n-1}, \widetilde{\mathbf{v}}\rangle, \quad \forall \widetilde{\mathbf{v}} \in \mathbf{V}_0(\Omega). \tag{4.46}$$

Taking $\widetilde{\mathbf{v}} = \widetilde{\mathbf{w}}^n$ in (4.46) yields

$$\|\widetilde{\mathbf{w}}^n\|^2 + 2\Delta t \nu \|\nabla \widetilde{\mathbf{w}}^n\|^2 \leq \|F^{n-1}\|^2 \tag{4.47}$$

and by using Lemma 3.1 and Poincaré inequality in (4.47), estimate (4.44) is obtained.

From (4.24) we deduce

$$\|\mathbf{v}^n\|^2 = \|\mathbf{w}\|^2 \alpha_n^2 + 2\langle \widetilde{\mathbf{w}}^n, \mathbf{w}\rangle \alpha_n + \|\widetilde{\mathbf{w}}^n\|^2,$$

and the polynomial $P(\alpha_n)$ of degree two with real coefficients is considered

$$P(\alpha_n) = \|\mathbf{v}^n\|^2 - \|\mathbf{v}^{n-1}\|^2 = \|\mathbf{w}\|^2 \alpha_n^2 + 2\langle \widetilde{\mathbf{w}}^n, \mathbf{w}\rangle \alpha_n + \|\widetilde{\mathbf{w}}^n\|^2 - \theta^2 \|\mathbf{v}^{n-1}\|^2. \tag{4.48}$$

Consequently, we have $\|\widetilde{\mathbf{w}}^n\|^2 - \theta^2\|\mathbf{v}^{n-1}\|^2 \leq 0$ from (4.44) and since $\|\mathbf{w}\|^2 > 0$, P has two solutions $\alpha_{n_1} \leq 0$ and $\alpha_{n_2} \geq 0$. For all $\alpha_n \in [\alpha_{n_1}, \alpha_{n_2}]$ we have $P(\alpha_n) \leq 0$ and hence, $\|\mathbf{v}^n\| \leq \|\mathbf{v}_0\| \exp\left(-\mu t^n\right)$. □

4 Numerical simulations

In this section, numerical simulations are performed in order to validate the theoretical results obtained in the previous sections. As in [36], two-dimensional test cases are considered by simulating the flow around a cylinder with circular cross-section.

4.1 Finite-element variational formulations

A weak formulation and a mixed Galerkin finite-element method are used to approximate the stationary problem (4.1) and the stabilization systems govern by (4.25) and (4.26). The spaces $\mathbf{V}_\phi$ and W_0 are introduced

$$\begin{aligned}\mathbf{V}_\phi(\Omega) &= \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{u} = 0 \text{ on } \Gamma_l, \mathbf{u} = \boldsymbol{\phi} \text{ on } \Gamma_b\},\\ \mathbf{W}_0(\Omega) &= \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{u} = 0 \text{ on } \Gamma_b \cup \Gamma_l\}.\end{aligned}$$

and L_0^2 is the pressure space with zero mean value

$$L_0^2(\Omega) = \left\{ p \in L^2(\Omega),\ \int_\Omega p(\mathbf{x})\, d\mathbf{x} = 0 \right\}.$$

Let $\mathcal{T}_h$ a standard finite-element triangulation of Ω with h being the maximal length of the edges of $\mathcal{T}_h$ and $\boldsymbol{\phi} \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$. The spaces $\mathbf{V}_\phi^h$, $\mathbf{W}_0^h$, $\mathbf{U}_0^h$ and S_0^h are the discrete counterpart of V_ϕ, $\mathbf{W}_0$, $\mathbf{U}_0$ and L_0^2, respectively, and we have $\mathbf{V}_\phi^h \subset \mathbf{V}_\phi(\Omega)$, $\mathbf{W}_0^h \subset \mathbf{W}_0(\Omega)$, $\mathbf{U}_0^h \subset \mathbf{H}_0^1(\Omega)$ and $S_0^h \subset L_0^2(\Omega)$.

The Galerkin formulation of the problem is defined as follows

(i) For (4.1), and $k = 1, 2, 3, \cdots$, find $\mathbf{v}_h^k \in \mathbf{V}_{v_b}^h$ and $p_h^k \in S_0^h$ such that

$$\begin{cases} (a) \quad \nu a(\mathbf{v}_h^{(k)}, \widetilde{\mathbf{v}}_h) + \sigma c(\mathbf{v}_h^{(k)}; \mathbf{v}_h^{(k-1)}, \widetilde{\mathbf{v}}_h) + c(\mathbf{v}_h^{(k-1)}; \mathbf{v}_h^{(k)}, \widetilde{\mathbf{v}}_h) + b(\widetilde{\mathbf{v}}_h, p_h^{(k)}) \\ \qquad\qquad = \sigma c(\mathbf{v}_h^{(k-1)}; \mathbf{v}_h^{(k-1)}, \widetilde{\mathbf{v}}_h) \\ (b) \quad b(\mathbf{v}_h^k, \pi_h) = 0, \end{cases} \tag{4.49}$$

$\forall (\widetilde{\mathbf{v}}_h, \pi_h) \in \mathbf{U}_0^h \times S_0^h$.

Given the velocity $\mathbf{v}_h^0$ and an integer m, one can generate the sequence $(\mathbf{v}_h^k, p_h^k)$ ($k = 1, 2, \cdots$) by solving the linear problem (4.49) with $\sigma = 0$ for $k \leq m$ and $\sigma = 1$ for $k > m$. The algorithm terminates when the maximum value of $\|\mathbf{v}_h^{(k)} - \mathbf{v}_h^{(k-1)}\| / \|\mathbf{v}_h^{(k)}\|$ is less or equal to ϵ, where ϵ is the prescribed tolerance.

(ii) For (4.25), find $\mathbf{w}_h \in \mathbf{V}_g^h$ and $q_h \in S_0^h$ such that

$$\begin{cases} (a) \quad \frac{1}{\Delta t}\langle \mathbf{w}_h, \widetilde{\mathbf{v}}_h \rangle + \nu a(\mathbf{w}_h, \widetilde{\mathbf{v}}_h) + b(\widetilde{\mathbf{v}}_h, q_h) = 0, \\ (b) \quad b(\mathbf{w}_h, \pi_h) = 0, \end{cases} \tag{4.50}$$

$\forall (\widetilde{\mathbf{v}}_h, \pi_h) \in \mathbf{U}_0^h \times S_0^h$.

(iii) For (4.26), find $\widetilde{\mathbf{w}}_h^n \in \mathbf{W}_0^h$ and $\widetilde{q}_h^n \in S_0^h$ such that

$$\begin{cases} (a) & \dfrac{1}{\Delta t}\langle \widetilde{\mathbf{w}}_h^n, \widetilde{\mathbf{v}}_h\rangle + \nu a(\widetilde{\mathbf{w}}_h^n, \widetilde{\mathbf{v}}_h) + b(\widetilde{\mathbf{v}}_h, \widetilde{q}_h^n) = \dfrac{1}{\Delta t}\langle F^{n-1}, \widetilde{\mathbf{v}}_h\rangle, \\ (b) & b(\widetilde{\mathbf{w}}_h^n, \pi_h) = 0, \end{cases} \tag{4.51}$$

$\forall(\widetilde{\mathbf{v}}_h, \pi_h) \in \mathbf{U}_0^h \times S_0^h$.

4.2 Geometry and parameters of the model

The geometry of the channel with an obstacle is described in Figure 4.1. As in [36], we consider a rectangular domain $\Omega = [0, 2.2 \text{ m}] \times [0, H]$ with a disk of diameter $D = 0.1$ m and centered at point $(0.2, 0.2)$. For a channel height $H = 0.4$ m, the inflow condition imposed at the bottom $\Gamma_e = \{0\} \times [0, H]$, is a parabolic flow defined by

$$v_1(0, x_2) = 4v_\infty \frac{x_2}{H}\left(1 - \frac{x_2}{H}\right), \qquad v_2 = 0. \tag{4.52}$$

The Reynolds number is then defined by $R_e = \frac{DU_0}{\nu}$ with the mean velocity

$$U_0(t) = 2v_1(t; 0, H/2)/3.$$

On Γ_l, defined by the top and bottom parts of the channel, the no-slip conditions $v_1 = v_2 = 0$ are imposed. At the outflow boundary of the channel, located at $\Gamma_s = \{2.2\} \times [0, H]$, we take the natural boundary condition $\nu \nabla \mathbf{v}^n \cdot \mathbf{n} - p^n \mathbf{n} = 0$, that arises from the weak formulation. In the sequel, the kinematic viscosity is fixed as $\nu = 10^{-4} m^2/s$ and the time step $\Delta t = 10^{-3} s$.

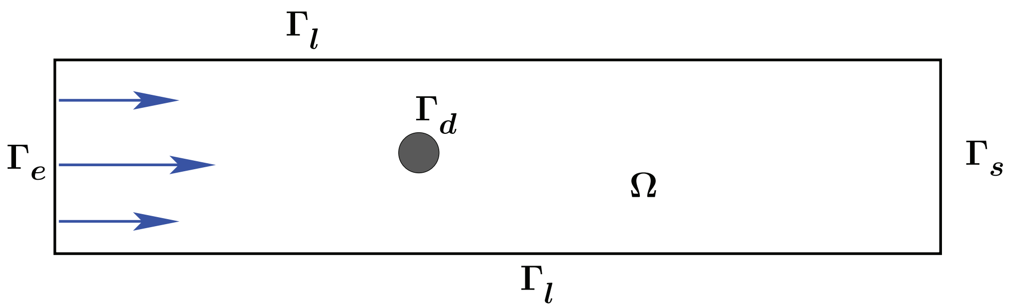

FIGURE 4.1 – Description of the domain Ω and of the four connected components Γ_e, Γ_l, Γ_d and Γ_s.

4.3 Numerical tests

Accurate mixed Galerkin finite-element computations are obtained using the $P_2 - P_1$ Taylor-Hood finite element pair [15, 27], with

$$\begin{aligned}
\mathbf{V}_\phi^h &= \{\mathbf{v}_h \mid \mathbf{v}_h \in C^0(\bar\Omega),\ \ \mathbf{v}_h|_T \in (P_2)^2,\ \ \forall T \in \mathcal{T}_h;\ \ \mathbf{v}_h = 0 \text{ on } \Gamma_l;\ \ \mathbf{v}_h = \boldsymbol{\phi}_h \text{ on } \Gamma_b = \Gamma_e \cup \Gamma_d\},\\
\mathbf{W}_0^h &= \{\mathbf{v}_h \mid \mathbf{v}_h \in C^0(\bar\Omega),\ \ \mathbf{v}_h|_T \in (P_2)^2,\ \ \forall T \in \mathcal{T}_h;\ \ \mathbf{v}_h = 0 \text{ on } \Gamma_l \cup \Gamma_e \cup \Gamma_d\},\\
\mathbf{U}_0^h &= \{\mathbf{v}_h \mid \mathbf{v}_h \in C^0(\bar\Omega),\ \ \mathbf{v}_h|_T \in (P_2)^2,\ \ \forall T \in \mathcal{T}_h;\ \ \mathbf{v}_h = 0 \text{ on } \Gamma\},\\
S_0^h &= \{q_h \mid q_h \in C^0(\bar\Omega),\ \ q_h|_T \in P_1,\ \ \forall T \in \mathcal{T}_h;\ \ \int_\Omega q_h = 0\},
\end{aligned}$$

where P_k is the space of the polynomials of degree $\le k$, expressed in terms of $\mathbf{x} = (x_1, x_2)$.

4.3.1 Test 1 : Control on Γ_e with $R_e = 500$

In the first test, the control is built on Γ_e, namely at the entrance boundary. The steady-state $(\mathbf{v}_s, q_s)$, shown in Figure 4.2, is obtained by solving (4.49) with $\mathbf{v}_s = 0$ on $\Gamma_d \cup \Gamma_l$, $\mathbf{v}_s = (v_1, v_2)$ on Γ_e with $v_\infty = 0.75$ m/s in (4.52), yielding the Reynolds number $R_e = 500$. Such a steady-state $(\mathbf{v}_s, q_s)$ is employed as an initial condition to solve for the Navier-Stokes system with $R_e = 1000$, i.e. using $v_\infty = 1.5$ m/s in (4.52). The solution obtained at $t = 5$ s, and shown in Figure 4.3, is not symmetrical along the axis $y = H/2$, and this behavior is due to the use of a large Reynolds number ($R_e = 1000$ in the experiment). The break in the symmetry can be explained by the influence of the various truncation and rounding errors that are present in the calculations. The perturbed solution in Figure 4.3 is then employed as an initial solution to solve for the control problem (4.22) with $R_e = 500$.

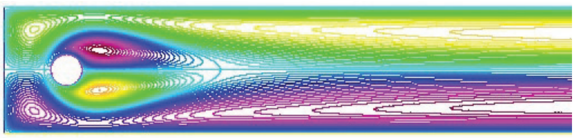

FIGURE 4.2 – Test 1 : Streamlines of the steady-state for $R_e = 500$.

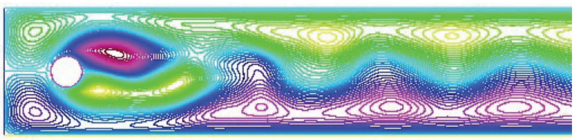

FIGURE 4.3 – Tests 1 and 2 : Streamlines of the initial velocity for $R_e = 1000$.

The control problem (4.22) is solved in three steps :

(i) Firstly, we search for $(\mathbf{w}_h, q_h)$ satisfying (4.25) with

$$\mathbf{w}_h = 0 \;\text{ on } \Gamma_l \cup \Gamma_d, \quad \nu \nabla \mathbf{w}_h \cdot \mathbf{n} - q_h \mathbf{n} = 0 \;\text{ on } \Gamma_s \quad \text{and} \quad \mathbf{w}_h = (v_1, v_2) \;\text{ on } \Gamma_e,$$

where (v_1, v_2) satisfies (4.52) with $v_\infty = 0.3$ m/s as the starting velocity, namely g in (4.22-d).

(ii) Secondly, at each time step we search for $(\widetilde{\mathbf{w}}_h^n, \widetilde{q}_h{}^n)$ satisfying (4.26) with

$$\widetilde{\mathbf{w}}_h^n = 0 \;\text{ on } \Gamma_l \cup \Gamma_d \cup \Gamma_e \quad \text{and} \quad \nu \nabla \widetilde{\mathbf{w}}_h{}^n \cdot \mathbf{n} - \widetilde{q}_h{}^n \mathbf{n} = 0 \;\text{ on } \Gamma_s.$$

(iii) Finally, we search for the solution $(\mathbf{v}_h^n, p_h{}^n)$ of (4.22) such that

$$\begin{cases} \mathbf{v}_h^n &= \widetilde{\mathbf{w}}_h^n + \alpha_n \mathbf{w}_h \\ p_h^n &= \widetilde{q}_h^n + \alpha_n q_h, \end{cases} \tag{4.53}$$

where the control $\alpha_n,\ n = 1, 2, 3, \cdots$, is chosen such that

$$\alpha_n^2 \|\mathbf{w}_h\|^2 + 2\alpha_n \langle \mathbf{w}_h, \widetilde{\mathbf{w}}_h^n \rangle + \|\widetilde{\mathbf{w}}_h^n\|^2 \leq \|\mathbf{v}_h^{n-1}\|^2. \tag{4.54}$$

To obtain α_n, we take

$$A = \|\mathbf{w}_h\|^2, \quad B_n = 2\langle \mathbf{w}_h, \widetilde{\mathbf{w}}_h^n \rangle, \quad C_n = \|\widetilde{\mathbf{w}}_h^n\|^2 - \|\mathbf{v}_h^{n-1}\|^2, \quad \Delta_n = B_n^2 - 4 \times A_n \times C_n.$$

According to (4.44) we have $\Delta_n > 0$, and consequently, for all constant $K > 2$, we obtain

$$\alpha_n^\pm = \frac{-B_n \pm \sqrt{\Delta_n}}{K \times A}. \tag{4.55}$$

Figure 4.4 shows the energy and the control evolution (α_n^+) in time for $K = 2.01$ (the red curve) and $K = 4.01$ (the blue curve). As the values of K increase, the energy decreases and α_n^+ tends to zero, as expected. The quantities α_n^+ and α_n^- correspond to inflow and outflow conditions, respectively, for the control problem (and not for the Navier-Stokes system). The choice α_n^+ is consistent with $\alpha_n \in [\alpha_{n_1}, \alpha_{n_2}]$, with $\alpha_{n_1} \leq 0$ and $\alpha_{n_2} \geq 0$, in (4.48). The first component (along the x_1-axis) of $\boldsymbol{\psi} = \mathbf{v}_s + \mathbf{v}$, with $K = 2.01$ is also displayed in Figure 4.8 at different times of the simulation. We observe that the system is progressively stabilizing towards an equilibrium steady state.

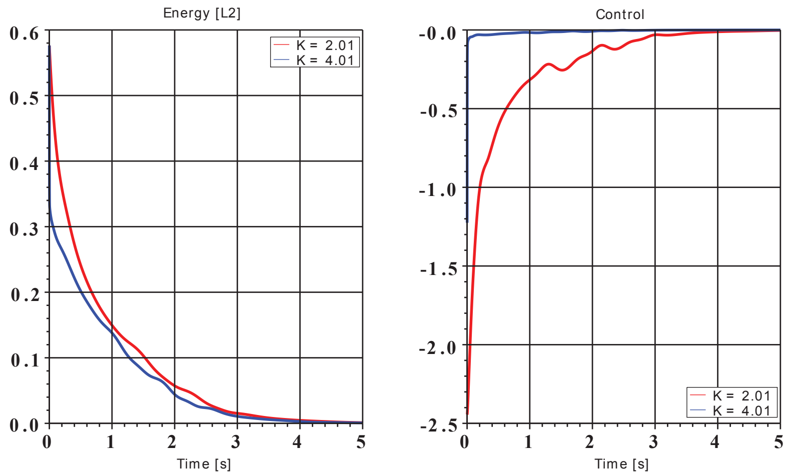

FIGURE 4.4 – Test 1 : Energy and control evolution (α_n^+) in time for $K = 2.01$ (the red curve) and $K = 4.01$ (the blue curve).

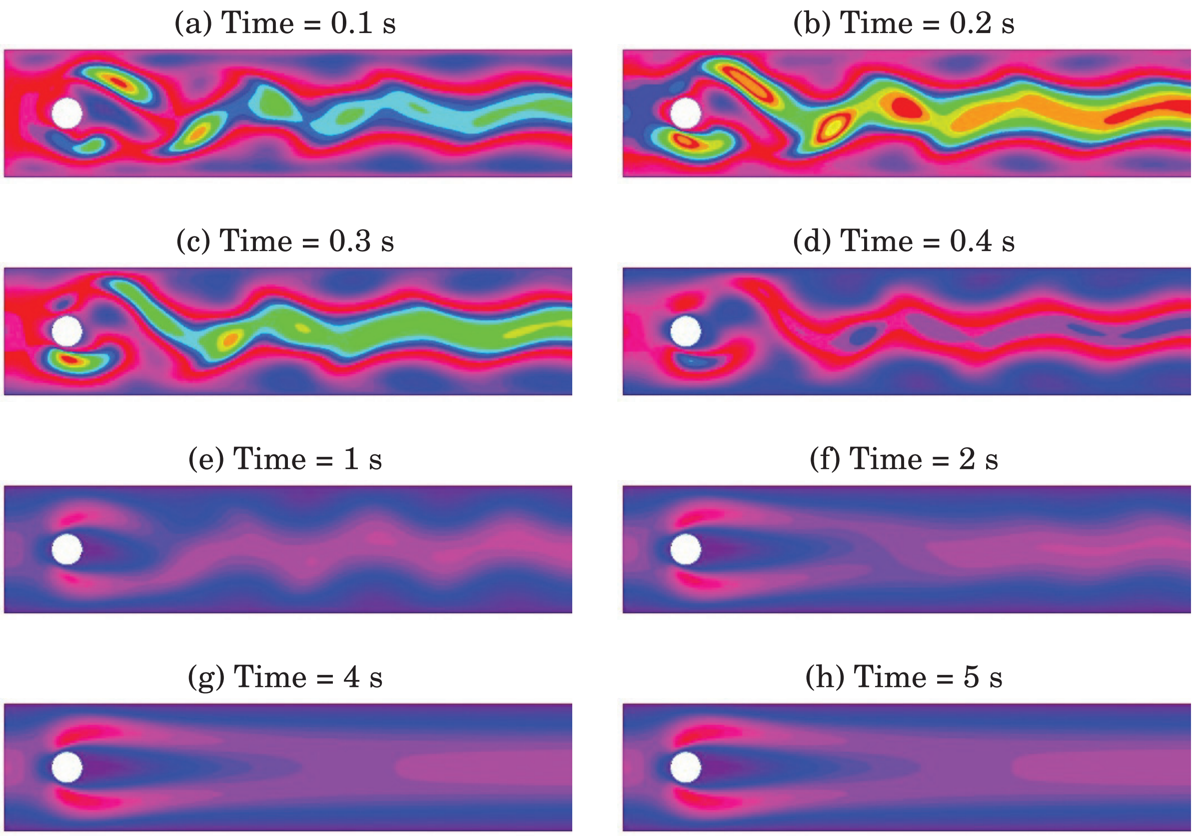

FIGURE 4.5 – Test 1 : The first component (along the x_1-axis) of $\psi = \mathrm{v}_s + \mathrm{v}$, with $K = 2.01$ at different times of the simulation.

4.3.2 Test 2 : Control around a part of Γ_d with $R_e = 1000$

This section is devoted to the suppression of vortex shedding past a cylinder. Our goal is to control the Navier-Stokes system, but instead of building the control at the entrance boundary Γ_e, the control is built on a part of Γ_d (the right section). Compared to Test 1, the difference is to try to stabilize the solution around the steady state $\mathbf{v}_s$ at $R_e = 1000$, by starting from an initial perturbation, also obtained at $R_e = 1000$. The initial perturbation is the same as for Test 1, and it is shown in Figure 4.3. As for Test 1, the steady state $\mathbf{v}_s$ is obtained by solving (4.49), with $\mathbf{v}_s = 0$ on Γ_d and $v_\infty = 1.5$ m/s in (4.52), yielding $R_e = 1000$. The first component (along the x_1-axis) of $\mathbf{v}_s$ is displayed in Figure 4.6 for $R_e = 1000$.

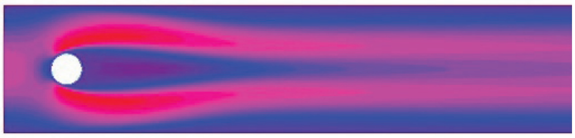

FIGURE 4.6 – Test 2 : The first component (along the x_1-axis) of the steady-state velocity $\mathbf{v}_s$ for $R_e = 1000$.

The control build on the right part of Γ_d is a suction-blowing action normal to the boundary of the disk on the two slots $\mathcal{C}_d^1 = [0.21, 0.25] \times [0.0, 0.2[$ and $\mathcal{C}_d^2 = [0.21, 0.25] \times]0.2, 0.4]$, symmetrical with respect to the axis $x_2 = 0.2$. To solve for the stabilization problem (4.22), we let $D(\mathbf{v}, p) = \nu \nabla \mathbf{v} \cdot \mathbf{n} - p\mathbf{n}$ and we denote by $\mathbf{N}^i(\mathbf{x}),\ i = 1, 2$, the outward normal unit vector to $\mathcal{C}_d^i$.

The problem is again solved in three steps :

(i) Firstly, we search for $(\mathbf{w}_h^1, q_h^1)$ satisfying (4.25) with :

$$\mathbf{w}_h^1 = 0 \ \text{ on } \Gamma_l \cup \Gamma_e \cup (\Gamma_d \backslash \mathcal{C}_d^1),\ D(\mathbf{w}_h^1, q_h^1) = 0 \ \text{ on } \Gamma_s,\ \mathbf{w}_h^1 = 0.01 \times \mathbf{N}^1(\mathbf{x}) \ \text{ on } \mathcal{C}_d^1,$$

and $(\mathbf{w}_h^2, q_h^2)$ satisfies (4.25) with :

$$\mathbf{w}_h^2 = 0 \ \text{ on } \Gamma_l \cup \Gamma_e \cup (\Gamma_d \backslash \mathcal{C}_d^2),\ D(\mathbf{w}_h^2, q_h^2) = 0 \ \text{ on } \Gamma_s,\ \mathbf{w}_h^2 = 0.01 \times \mathbf{N}^2(\mathbf{x}) \ \text{ on } \mathcal{C}_d^2.$$

(ii) Secondly, at each time step we search for $(\widetilde{\mathbf{w}}_h^n, \widetilde{q}_h^{\,n})$ satisfying (4.26) with :

$$\widetilde{\mathbf{w}}_h^n = 0 \ \text{ on } \Gamma_l \cup \Gamma_d \cup \Gamma_e,\ D(\widetilde{\mathbf{w}}_h^{\,n}, \widetilde{q}_h^{\,n}) = 0 \ \text{ on } \Gamma_s.$$

Due to the symmetry breaking, the pressure force exerted on the boundary $\mathcal{C}_d^1$ and $\mathcal{C}_d^2$ is disproportionate. To restore this imbalance, we search for β_n such that

$$\beta_n B_1(\widetilde{\mathbf{w}}_h^{\,n}, \widetilde{q}_h^{\,n}; \mathbf{w}_h^1) = B_2(\widetilde{\mathbf{w}}_h^{\,n}, \widetilde{q}_h^{\,n}; \mathbf{w}_h^2), \tag{4.56}$$

where $B_i(\widetilde{\mathbf{w}}_h{}^n, \widetilde{q}_h{}^n; \mathbf{w}_h^i) = \displaystyle\int_{\mathcal{C}_d^i} [\nu \nabla \widetilde{\mathbf{w}}_h{}^n \cdot \mathbf{n} - \widetilde{q}_h{}^n \mathbf{n}] \cdot \mathbf{w}_h^i$.

By taking $(\mathbf{w}_h^n, q_h^n) = (\beta_n \mathbf{w}_h^1 + \mathbf{w}_h^2, \beta_n q_h^1 + q_h^2)$ which satisfies (4.25) and according to (4.56), we have $B_1(\widetilde{\mathbf{w}}_h{}^n, \widetilde{q}_h{}^n; \mathbf{w}_h^n) = B_2(\widetilde{\mathbf{w}}_h{}^n, \widetilde{q}_h{}^n; \mathbf{w}_h^n)$.

(iii) Finally, we search for $(\mathbf{v}_h^n, p_h{}^n)$ of (4.22) such that

$$\begin{cases} \mathbf{v}_h^n & = \widetilde{\mathbf{w}}_h^n + \alpha_n \mathbf{w}_h^n \\ p_h^n & = \widetilde{q}_h^n + \alpha_n q_h^n, \end{cases} \tag{4.57}$$

where the control $\alpha_n,\ n = 1, 2, 3, \cdots$, is chosen such that

$$\alpha_n^2 \|\mathbf{w}_h^n\|^2 + 2\alpha_n \langle \mathbf{w}_h^n, \widetilde{\mathbf{w}}_h^n \rangle + \|\widetilde{\mathbf{w}}_h^n\|^2 = \|\mathbf{v}_h^n\|^2 \leq C \|\mathbf{v}_h^{n-1}\|^2, \tag{4.58}$$

where the constant $C \geq 1$.

Note that for the uncontrolled case, $\alpha_n = 0$ for all $n > 0$, whereas for the controlled case, we take

$$A_n = \|\mathbf{w}_h^n\|^2, \quad B_n = 2\langle \mathbf{w}_h^n, \widetilde{\mathbf{w}}_h^n \rangle, \quad C_n = \|\widetilde{\mathbf{w}}_h^n\|^2 - \|\mathbf{v}_h^{n-1}\|^2, \quad \Delta_n = B_n^2 - 4 \times A_n \times C_n.$$

According to (4.44) we have $\Delta_n > 0$, and consequently, we search the control α_n as follows :

$$\alpha_n^{\pm} = \begin{cases} \dfrac{-B_n \pm \sqrt{\Delta_n}}{1.99 \times A_n} & \text{if } n < Iter = 100,\ 200 \text{ or } 300, \\ \dfrac{-B_n \pm \sqrt{\Delta_n}}{2.01 \times A_n} & \text{otherwise .} \end{cases} \tag{4.59}$$

Note that $\|\mathbf{v}_h^n\| > \|\mathbf{v}_h^{n-1}\|$ if $n < Iter$ and $\|\mathbf{v}_h^n\| \leq \|\mathbf{v}_h^{n-1}\|$ if $n \geq Iter$.

Figure 4.7 shows the energy and the control evolution (α_n^+) in time for different values of $Iter$. As the number of iterations increases, the energy stabilizes over time, and during the stabilization process, the lower values of energy are obtained for the higher values of $Iter$. However, shortly after the beginning of the simulation, pics of energy are observed before the stabilization process and the higher values of energy are reached for the higher values of $Iter$. The origin of the pics is due to the choice of α_n^+ at the early times of the simulation. Indeed, we purposely choose $\alpha_n^+ < \alpha_{n_1}$ or $\alpha_n^+ > \alpha_{n_2}$, in order to avoid the presence of propagating eddies close to the right part of Γ_d. As soon as the control area is free of such eddies the choice $\alpha_n \in [\alpha_{n_1}, \alpha_{n_2}]$, is imposed and the stabilizing energy process can take place. The switch in α_n^+ is done at time 0.3 s for $Iter = 300$ when the energy reaches its maximum value. As the control α_n^+ weakens and is close to zero, the energy progressively stabilizes around an unstable state. Note that the pics in energy, observed at the beginning of the simulations are not present for low Reynolds numbers, namely $R_e = 500$, and in that case the stabilization process converges around a steady-state. This suggests that Test 2 is a very challenging test case. The remark made about α_n^- for Test 1 in Section 4.3.1 is still valid. The first component (along the x_1-axis) of $\boldsymbol{\psi} = \mathbf{v}_s + \mathbf{v}$, with $Iter = 300$ is also displayed in Figure 4.8 at different times of the simulation. We observe that the system is progressively stabilizing towards an equilibrium steady state up to 2 s. After the steady state is reached, the eddy process starts to propagate again, as observed in Figure 4.8, since the control vanishes.

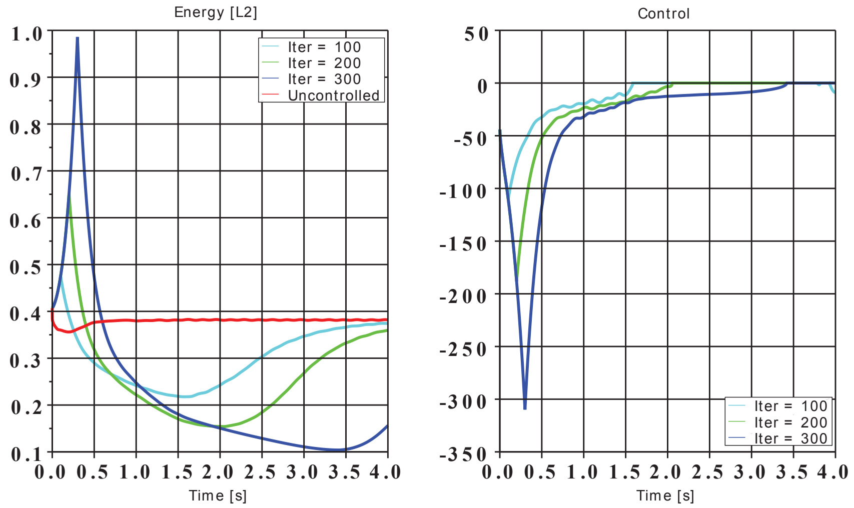

FIGURE 4.7 – Test 2 : Energy and control evolution (α_n^+) in time for different values of $Iter$.

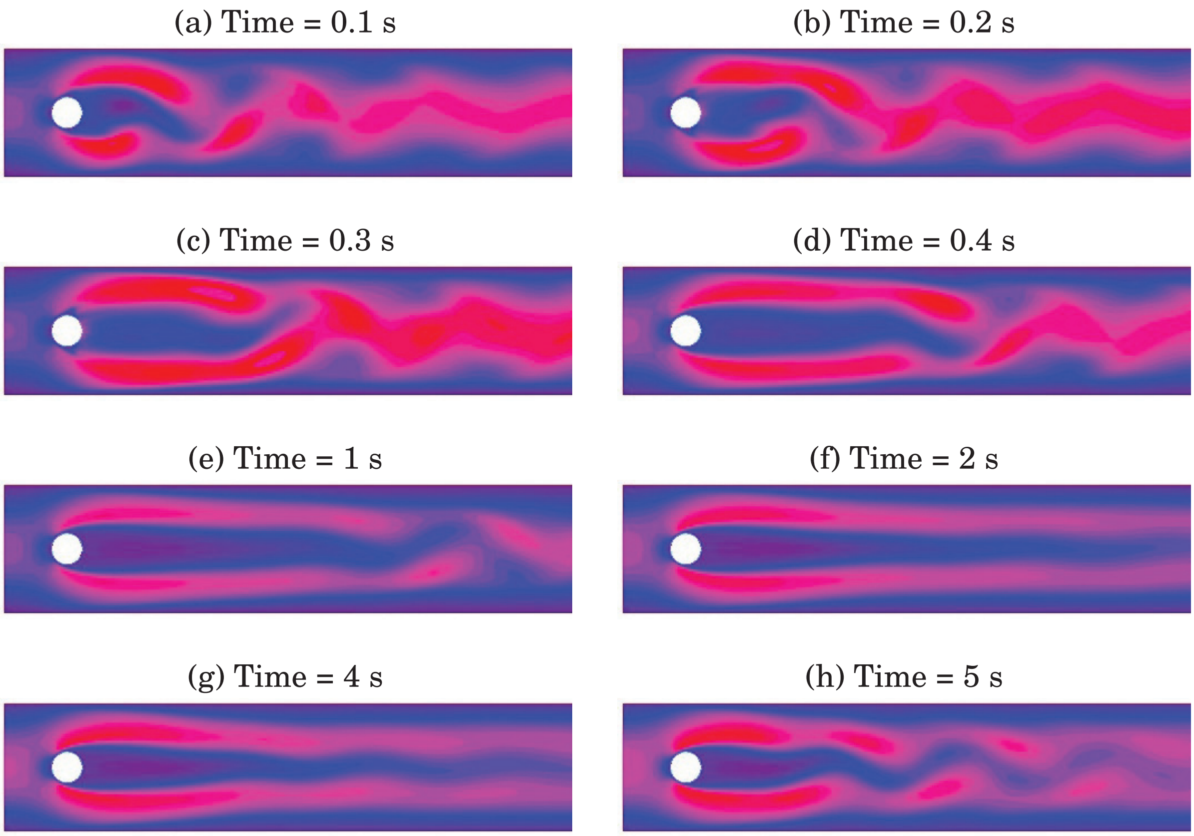

FIGURE 4.8 – Test 2 : The first component (along the x_1-axis) of $\psi = \mathrm{v}_s + \mathrm{v}$, with $Iter = 300$ at different times of the simulation.

5 Concluding remarks

In this paper, the numerical feedback stabilization of the two and three-dimensional Navier-Stokes equations in a bounded domain is studied around a given steady-state flow, using a boundary feedback control. In order to determine a feedback law, an extended system coupling the Navier-Stokes equations with an equation satisfied by the control on the domain boundary is considered. We first assume that on Σ_b (a part of the domain boundary), the trace of the fluid velocity is proportional to a given velocity profile g. The proportionality coefficient α measures the velocity flux at the interface. It is an unknown of the problem and is written in feedback form. By using the characteristic-Galerkin method, α is determined by solving a polynomial equation of degree one or two and the stabilizing boundary control is built such that the Dirichlet boundary control $\mathbf{v}_b = \alpha \mathrm{g}$ is satisfied on Σ_b. Numerical solutions of two test problems to simulate the boundary feedback control, by stabilizing the two-dimensional Navier-Stokes system around a circular obstacle, illustrate the theoretical results of the present paper. Such an approach appears to be promising.

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